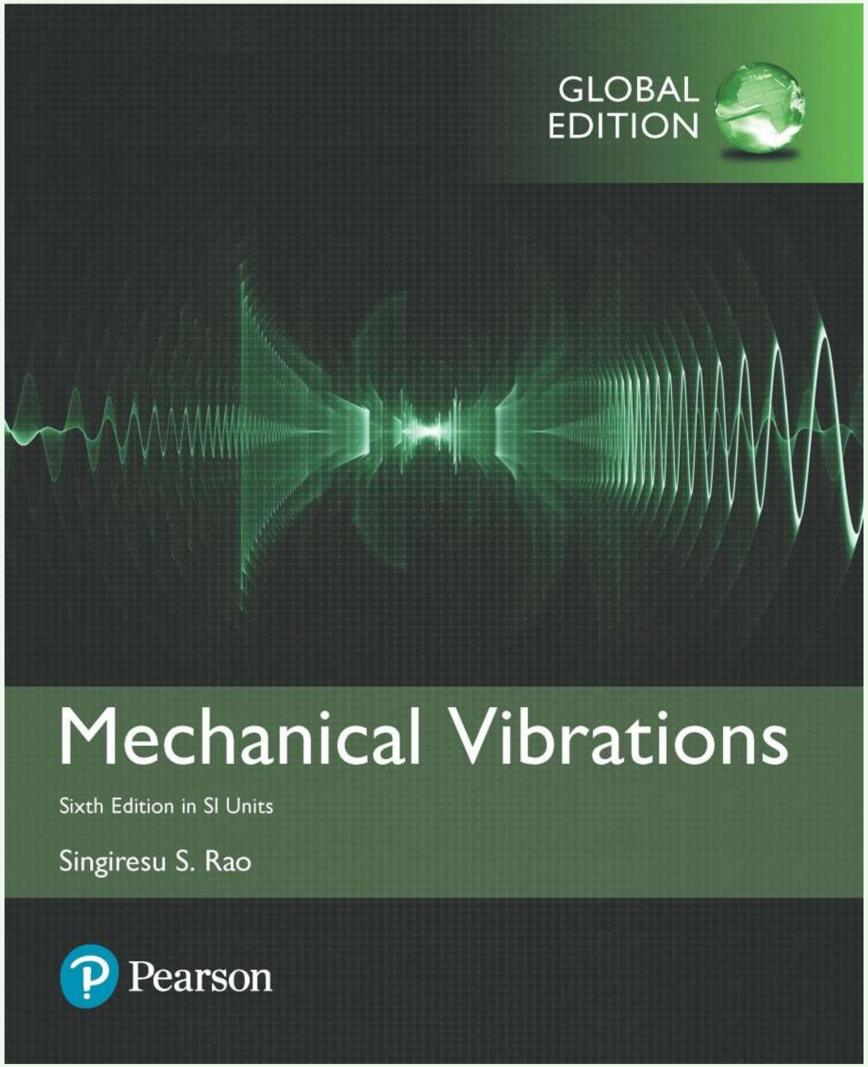
Mechanical Vibrations



Sixth Edition in SI Units

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Chapter 4 Vibration Under General Forcing Condition

4

Chapter Outline

4.1

Introduction

4.1

4.1 Introduction

- *Shock* is defined as the small forcing function or excitation as compared to the natural time period of the system.

- Some examples of general forcing functions include the motion imparted by a cam to the follower; the vibration felt by an instrument when its package is dropped from a height; etc.

- The transient response of a system can be found by using what is known as the *convolution integral*.

4.2
Response Under a General Periodic Force

4.2

4.2 Response Under a General Periodic Force

- The equation of motion can be expressed as

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \qquad (4.8)$$

- The steady-state solution of the equation is derived as:

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{(a_j / k)}{\sqrt{(1 - j^2 r^2)^2 + (2\zeta j r)^2}} \cos(j\omega t - \phi_j)$$
$$+ \sum_{j=1}^{\infty} \frac{(b_j / k)}{\sqrt{(1 - j^2 r^2)^2 + (2\zeta j r)^2}} \sin(j\omega t - \phi_j)$$

4.2 Response Under a General Periodic Force

Example 4.5
Total Response Under Harmonic Base Excitation

Find the total response of a viscously damped single degree of freedom system subjected to a harmonic base excitation for the following data:

$$m = 10\text{kg}, \quad c = 20\text{N-m/s}, \quad k = 4000\text{N/m},$$
$$y(t) = 0.05 \sin 5t \ \text{ m}, \quad x_0 = 0.02 \text{ m}, \quad \dot{x}_0 = 10 \text{ m/s}.$$

4.2 Response Under a General Periodic Force

Example 4.5
Total Response Under Harmonic Base Excitation
Solution

The equation of motion of the system is given by:

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY \sin \omega t + c\omega Y \cos \omega t \quad \text{(E.1)}$$

The steady-state response of the system can be expressed as

$$x_p(t) = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \left[\frac{a_1}{k} \cos(\omega t - \phi_1) + \frac{b_1}{k} \sin(\omega t - \phi_1) \right] \quad \text{(E.2)}$$

4.2 Response Under a General Periodic Force

Example 4.5

Total Response Under Harmonic Base Excitation

Solution

We have,

$$Y = 0.05\text{ m}, \omega = 5\text{ rad/s}, \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20\text{ rad/s},$$

$$r = \frac{\omega}{\omega_n} = \frac{5}{20} = 0.25,\ \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{(4000)(10)}} = 0.05,$$

$$\omega_d = \sqrt{1-\zeta^2}\omega_n = 19.975\text{ rad/s}$$

$$a_1 = c\omega Y = (20)(5)(0.05) = 5,\ b_1 = kY = (4000)(0.05) = 200,$$

$$\phi_1 = \tan^{-1}\left(\frac{2(0.05)(0.25)}{1-(0.25)^2}\right) = 0.02666\text{ rad}$$

$$\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \sqrt{(1-0.25^2)^2 + (2(0.05)(0.25))^2} = 0.937833$$

4.2 Response Under a General Periodic Force

Example 4.5

Total Response Under Harmonic Base Excitation

Solution

The solution of the homogeneous equation is given by:

$$x_h(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - c = X_0 e^{-t}\cos(19.975t - \phi_0) \qquad \text{(E.3)}$$

where X_0 and Φ_0 are unknown constants

The total solution can be expressed as the superposition of $x_h(t)$ and $x_p(t)$ as:

$$x(t) = X_0 e^{-t}\cos(19.975t - \phi_0) + \frac{1}{0.937833}\left[\frac{5}{4000}\cos(5t - \phi_1) + \frac{200}{4000}\sin(5t - \phi_1)\right]$$

$$= X_0 e^{-t}\cos(19.975t - \phi_0) + 0.0013333\cos(5t - 0.026666)$$

$$+ 0.053314\sin(5t - 0.026666) \qquad \text{(E.4)}$$

4.2 Response Under a General Periodic Force

Example 4.5
Total Response Under Harmonic Base Excitation
Solution

Using Eqs.(E.4) and (E.5), we find

$$x_0(t) = x(t=0) = 0.02 = X_0 \cos\phi_0 + 0.001333\cos(0.02666) - 0.053314\sin(0.02666)$$

$$X_0 \cos\phi_0 = 0.020088 \qquad \text{(E.6)}$$

$$\dot{x}_0 = \dot{x}(t=0) = 10 = -X_0\cos\phi_0 + 19.975 X_0 \sin\phi_0$$
$$+0.006665\sin(0.02666) + 0.266572\cos(0.02666)$$

$$-X_0\cos\phi_0 + 19.975\sin\phi_0 = 9.733345 \qquad \text{(E.7)}$$

4.2 Response Under a General Periodic Force

Example 4.5

Total Response Under Harmonic Base Excitation

Solution

The solution of (E.6) and (E.7) yields X_0=0.488695 and Φ_0=1.529683 rad.

Thus the total response of the mass under base excitation, in meters, is given by

$$x(t) = 0.488695e^{-t}\cos(19.975t - 1.529683) + 0.001333\cos(5t - 0.02666) + 0.053314\sin(5t - 0.02666) \quad \text{(E.8)}$$

4.3
Response Under a Periodic Force of Irregular Form

4.3

4.3 Response Under a Periodic Force of Irregular Form

- In some cases, the force acting on a system may be quite irregular and may be determined only experimentally.

- The application of trapezoidal rule gives:

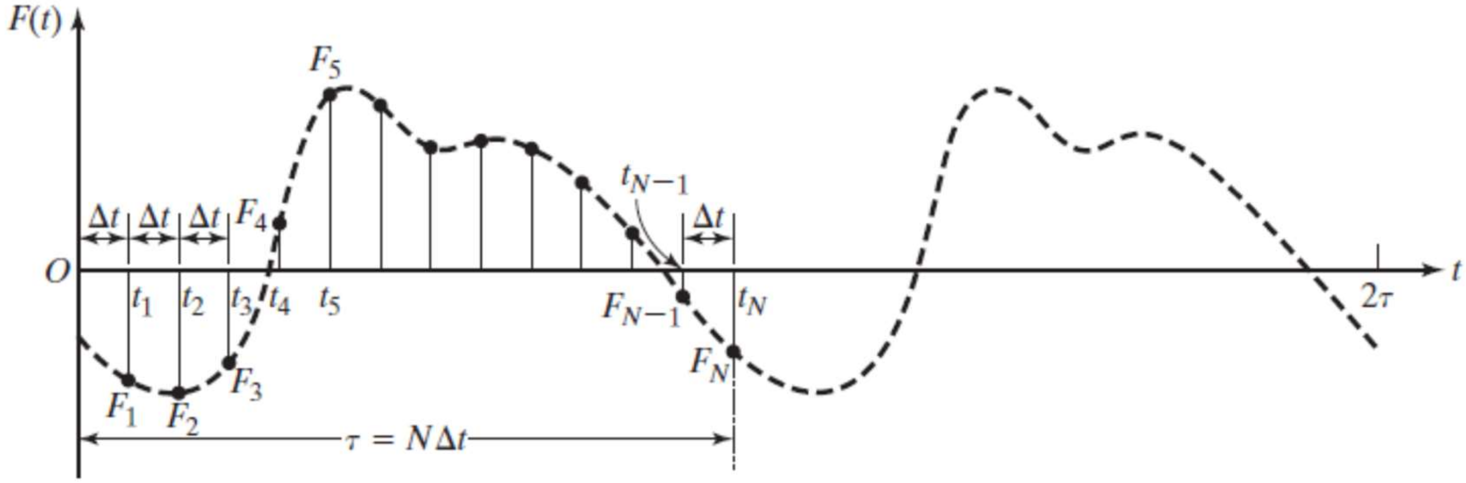
$$a_0 = \frac{2}{N}\sum_{i-1}^{N} F_i \qquad (4.9)$$

$$a_j = \frac{2}{N}\sum_{i-1}^{N} F_i \cos\frac{2j\pi t_i}{\tau}, \; j = 1,2,... \qquad (4.10)$$

$$b_j = \frac{2}{N}\sum_{i-1}^{N} F_i \sin\frac{2j\pi t_i}{\tau}, \; j = 1,2,... \qquad (4.11)$$

4.3 Response Under a Periodic Force of Irregular Form

- An irregular forcing function:



- Once the Fourier coefficients a_0, a_j, and b_j are known, the steady-state response of the system can be found using Eq.(4.13) with

$$r = \left(\frac{2\pi}{\tau\omega_n}\right)$$

4.3 Response Under a Periodic Force of Irregular Form

Example 4.6
Steady-State Vibration of a Hydraulic Valve

Find the steady-state response of the valve in the figure below if the pressure fluctuations in the chamber are found to be periodic. The valves of pressure measured at 0.01 second intervals in one cycle are given below.

Time, t_i (seconds)	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12
$p_i = p(t_i)$ (kN/m^2)	0	20	34	42	49	53	70	60	36	22	16	7	0

4.3 Response Under a Periodic Force of Irregular Form

Example 4.6
Steady-State Vibration of a Hydraulic Valve
Solution

Since the pressure fluctuations on the valve are periodic, the Fourier analysis of the given data of pressures in a cycle gives:

$$\begin{aligned} p(t) = 34083.3 - 26996.0 \cos 52.36t + 8307.7 \sin 52.36t \\ +1416.7 \cos 104.72t + 3608.3 \sin 104.72t \\ -5833.3 \cos 157.08t + 2333.3 \sin 157.08t + \ldots \text{ N/m}^2 \quad \text{(E.1)} \end{aligned}$$

Other quantities needed for the computation are

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.12} = 52.36 \text{ rad/s} \quad , \quad \omega_n = 100 \text{ rad/s} \quad , \quad r = \frac{\omega}{\omega_n} = 0.5236$$

4.3 Response Under a Periodic Force of Irregular Form

Example 4.6
Steady-State Vibration of a Hydraulic Valve
Solution

We have also

$$\zeta = 0.2$$

$$A = 0.000625\pi \text{ m}^2$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2\times 0.2\times 0.5236}{1-0.5236^2}\right) = 16.1°$$

$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}\left(\frac{4\times 0.2\times 0.5236}{1-4\times 0.5236^2}\right) = -77.01°$$

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}\left(\frac{6\times 0.2\times 0.5236}{1-9\times 0.5236^2}\right) = -23.18°$$

4.3 Response Under a Periodic Force of Irregular Form

Example 4.6
Steady-State Vibration of a Hydraulic Valve
Solution

The steady-state response of the valve can be expressed as

$$
\begin{aligned}
x_p(t) = \frac{34083.3A}{k} &- \frac{(26996.0A/k)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos(52.36t-\phi_1) + \frac{(8309.7A/k)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(52.36t-\phi_1) \\
&+ \frac{(1416.7A/k)}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\cos(104.72t-\phi_2) + \frac{(3608.3A/k)}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\sin(104.72t-\phi_2) \\
&- \frac{(5833.3A/k)}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\cos(157.08t-\phi_3) + \frac{(2333.3A/k)}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\sin(157.08t-\phi_3)
\end{aligned}
$$

4.4
Response Under a Nonperiodic Force

4.4

4.4 Response Under a Nonperiodic Force

- When the exciting force F(t) is nonperiodic, such as that due to the blast from an explosion, a different method of calculating the response is required.
- Various methods can be used to find the response of the system to an arbitrary excitation.
- Some of these methods are as follows:
  1. Representing the excitation by a Fourier integral
  2. Using the method of convolution integral
  3. Using the method of Laplace transforms
  4. First approximating F(t) by a suitable interpolation model and then using a numerical procedure
  5. Numerically integrating the equation of motion

4.5 Convolution Integral

4.5

4.5 Convolution Integral

- We have

$$\text{Impulse} = F\Delta t = m\dot{x}_2 - m\dot{x}_1 \qquad (4.12)$$

- By designating the magnitude of the impulse $F\Delta t$ by *F*, we can write, in general,

$$F = \int_t^{t+\Delta t} F dt \qquad (4.13)$$

- A unit impulse is defined as

$$f = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F dt = F dt = 1 \qquad (4.14)$$

4.5 Convolution Integral

- **Response to an impulse**

For an underdamped system, the solution of the equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (4.17)$$

is given by

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\} \qquad (4.18)$$

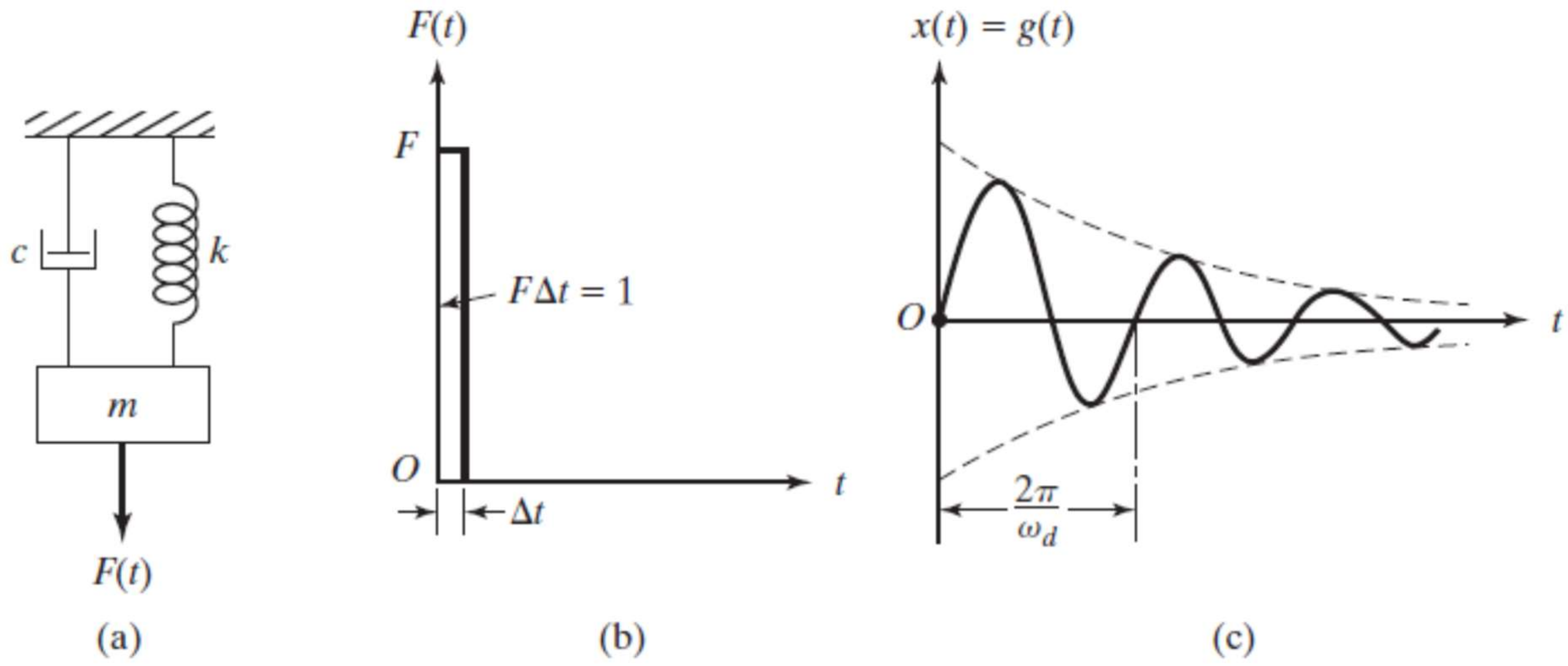
where $\zeta = \dfrac{c}{2m\omega_n}$ (4.19)

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{m}\right)^2} \qquad (4.20)$$

$$\omega_n = \sqrt{\frac{k}{m}} \qquad (4.21)$$

4.5 Convolution Integral

- **Response to an impulse**



A single-degree-of-freedom system subjected to an impulse

4.5 Convolution Integral

- **Response to an impulse**

If the mass is at rest before the unit impulse is applied, we obtain, from the impulse-momentum relation,

$$\text{Impulse} = f = 1 = m\dot{x}(t=0) - m\dot{x}(t=0^-) = m\dot{x}_0 \qquad (4.22)$$

Thus the initial conditions are given by

$$x(t=0) = x_0 = 0 \qquad (4.23)$$

$$\dot{x}(t=0) = \dot{x}_0 = \frac{1}{m} \qquad (4.24)$$

Hence, Eq.(4.18) reduces to

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t \qquad (4.25)$$

4.5 Convolution Integral

- **Response to an impulse**

If the magnitude of the impulse is *F* instead of unity, the initial velocity $\dot{x}_0$ is F/m and the response of the system becomes

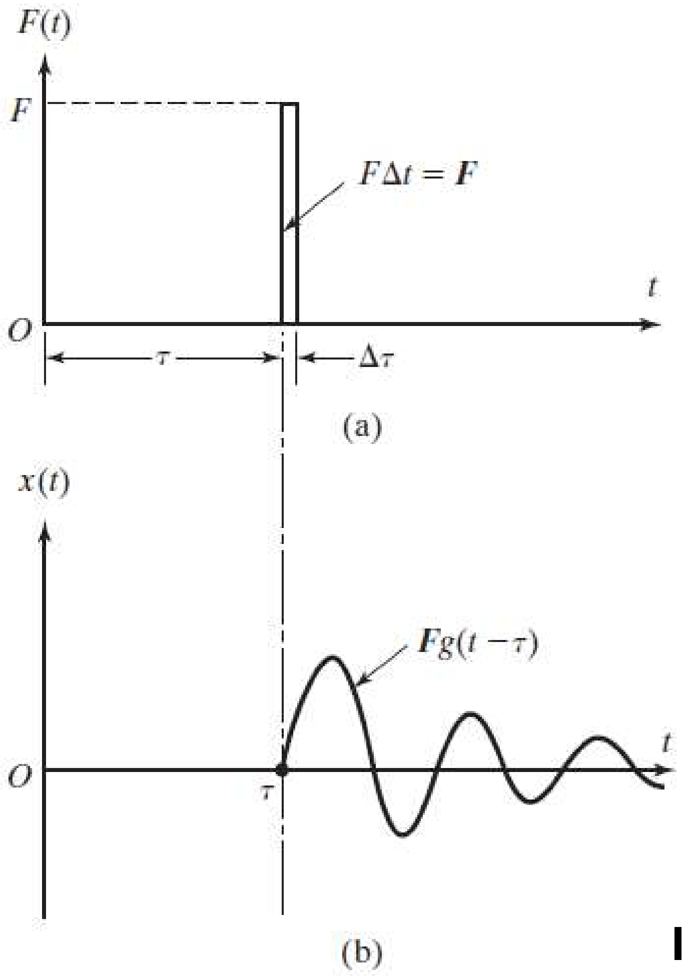
$$x(t) = \frac{Fe^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = Fg(t) \qquad (4.26)$$

If the impulse is applied at an arbitrary time $t = \tau$, it will change the velocity at $t = \tau$, shown in Fig.4.4(a). Thus,

$$x(t) = Fg(t-\tau) \qquad (4.27)$$

4.5 Convolution Integral

- **Response to an impulse**

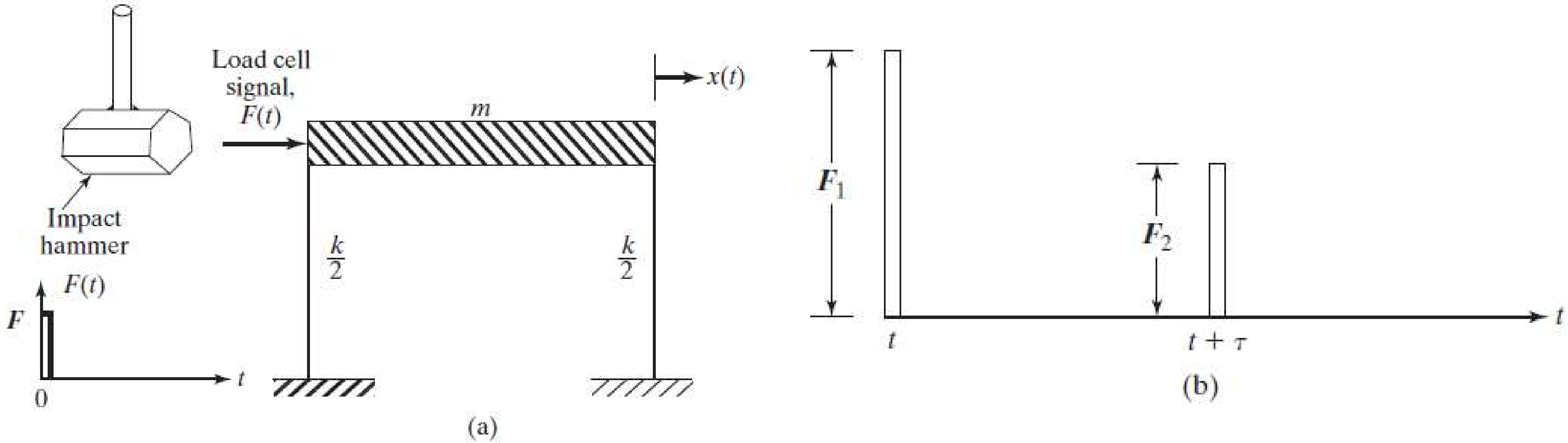


Impulse Response

4.5 Convolution Integral

Example 4.7
Response of a Structure Under Impact

In the vibration testing of a structure, an impact hammer with a load cell to measure the impact force is used to cause excitation, as shown in Fig.4.5(a). Assuming m = 5kg, k = 2000 N/m, c = 10 N-s/m and F = 20 N-s, find the response of the system.



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4.5 Convolution Integral

Example 4.7
Response of a Structure Under Impact
Solution

From the known data,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{5}} = 20 \text{ rad/s}, \quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{10}{2\sqrt{2000(5)}} = 0.05,$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 19.975 \text{ rad/s}$$

Assuming that the impact is given at $t = 0$, the response of the system

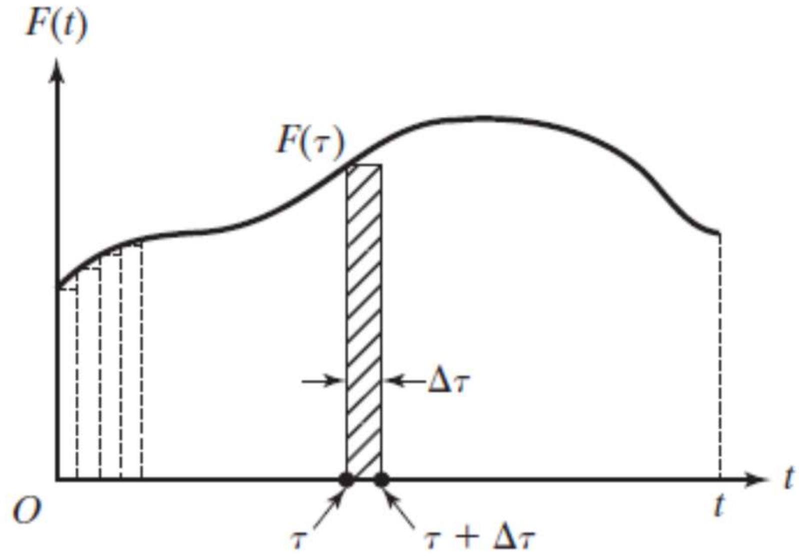
$$x_1(t) = F\frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = \frac{20}{(5)(19.975)}e^{-0.05(20)t}\sin 19.975t = 0.20025e^{-t}\sin 19.975t \text{ m} \qquad \text{(E.1)}$$

4.5 Convolution Integral

- **Response to General Forcing Condition**

Consider the response of the system under an arbitrary external force, the response is given by

$$\Delta x(t) = F(\tau)\Delta\tau g(t-\tau) \qquad (4.28)$$



An arbitrary (nonperiodic) forcing function

4.5 Convolution Integral

- **Response to General Forcing Condition**

The total response at time *t* can be found by summing all the responses due to the elementary impulses acting at all times τ :

$$x(t) \approx \sum F(\tau) g(t-\tau) \Delta\tau \qquad (4.29)$$

Letting $\Delta\tau \rightarrow 0$ and replacing the summation by integration, we obtain

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\zeta\omega_n (t-\tau)} \sin\omega_d (t-\tau) d\tau \qquad (4.31)$$

which is called the *convolution* or *Duhamel integral*

4.5 Convolution Integral

- **Response to Base Excitation**

For an undamped system subjected to base excitation, the relative displacement can be as

$$z(t) = -\frac{1}{\omega_d}\int_0^t \ddot{y}(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)d\tau \qquad (4.34)$$

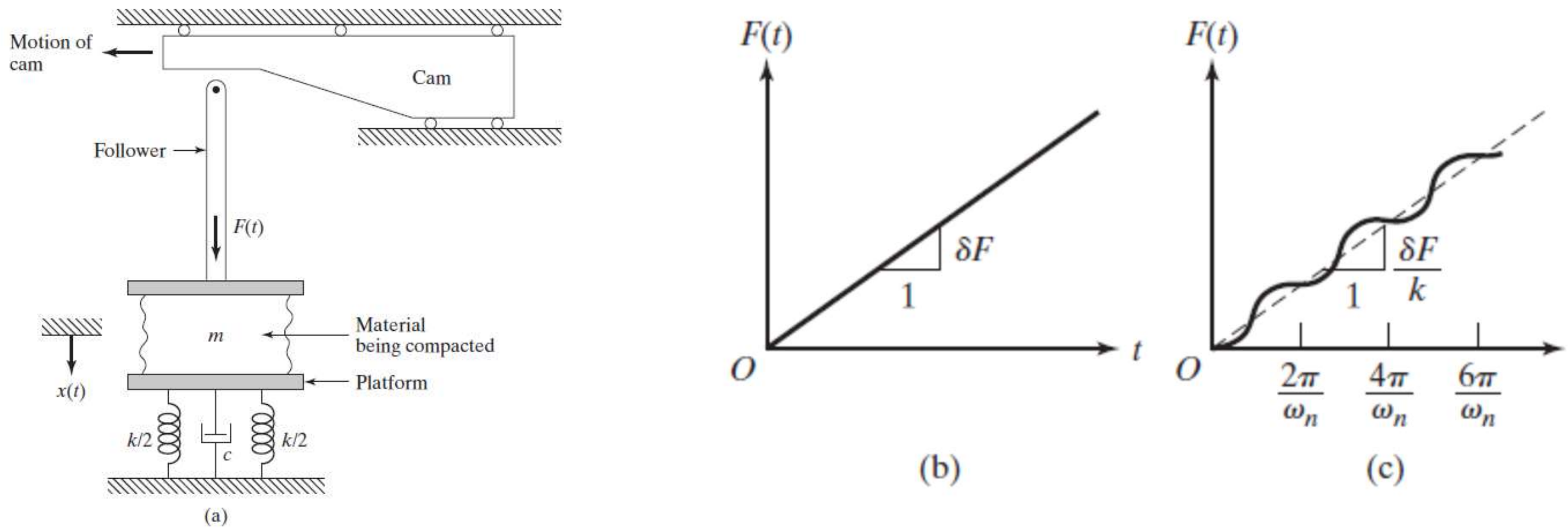
where the variable z replacing x

4.5 Convolution Integral

Example 4.12

Compacting Machine Under Linear Force

Determine the response of the compacting machine shown in Figure (a) when a linearly varying force (shown in Figure (b)) is applied due to the motion of the cam.



4.5 Convolution Integral

Example 4.12
Compacting Machine Under Linear Force
Solution

Figure (b) is known as the ramp function.

$$x(t) = \frac{\delta F}{m\omega_d}\int_0^t \tau e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau) d\tau$$

$$= \frac{\delta F}{m\omega_d}\int_0^t (t-\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau)(-d\tau)$$

$$- \frac{\delta F \cdot t}{m\omega_d}\int_0^t e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau)(-d\tau)$$

4.5 Convolution Integral

Example 4.12
Compacting Machine Under Linear Force
Solution

These integrals can be evaluated and the response expressed as follows:

$$x(t) = \frac{\delta F}{k}\left[t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t}\left(\frac{2\zeta}{\omega_n}\cos\omega_d t - \left\{\frac{\omega_d^2 - \zeta^2\omega_n^2}{\omega_n^2\omega_d}\right\}\sin\omega_d t\right)\right] \qquad \text{(E.1)}$$

For an undamped system, Eq.(E.1) reduces to

$$x(t) = \frac{\delta F}{\omega_n k}\left[\omega_n t - \sin\omega_n t\right] \qquad \text{(E.2)}$$

Fig. 4.13(c) shows the response given by Eq.(E.2)

4.6

Response Spectrum

4.6

4.6 Response Spectrum

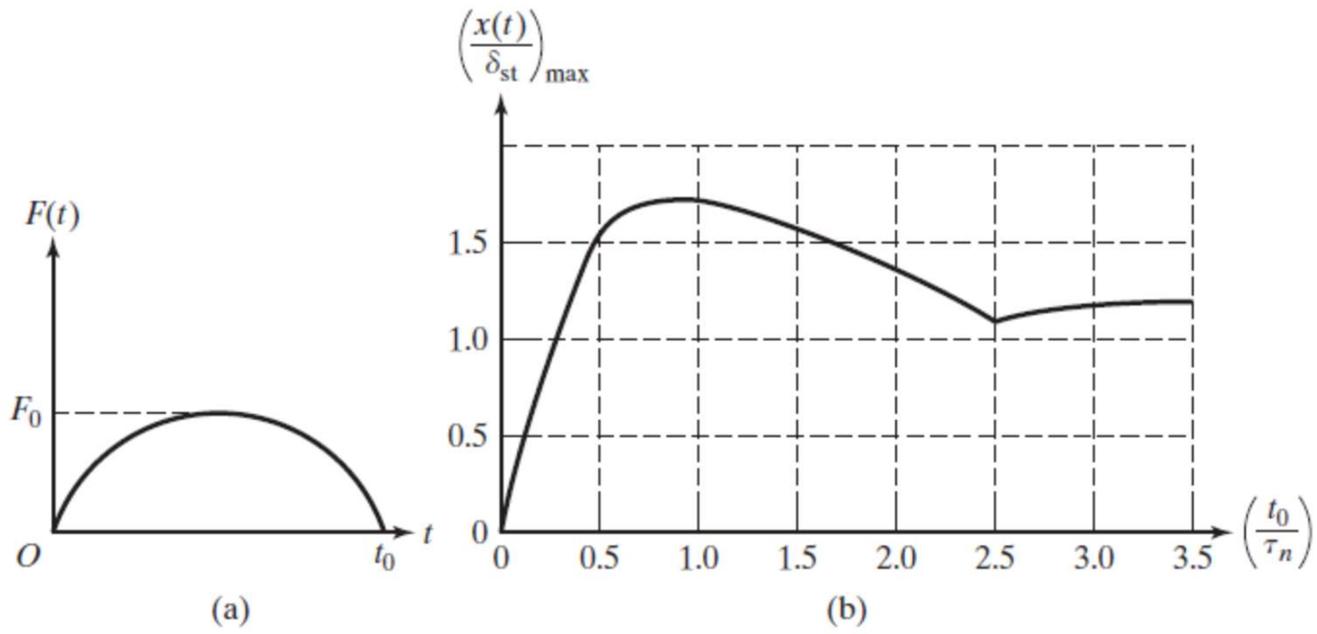
- The graph showing the variation of the maximum response (maximum displacement, velocity, acceleration, or any other quantity) with the natural frequency (or natural period) of a single degree of freedom system to a specified forcing function is known as the *response spectrum*.
- Example 4.14 illustrates the construction of a response spectrum.

4.6 Response Spectrum

Example 4.14

Response Spectrum of a Sinusoidal Pulse

Find the undamped response spectrum for the sinusoidal pulse force shown in the figure using the initial conditions



4.6 Response Spectrum

Example 4.14
Response Spectrum of a Sinusoidal Pulse
Solution

The equation of motion of an undamped system can be expressed as

$$m\ddot{x} + kx = F(t) = \begin{cases} F_0 \sin \omega t, & 0 \le t \le t_0 \\ 0, & t > t_0 \end{cases} \qquad \text{(E.1)}$$

$$\text{where } \omega = \frac{\pi}{t_0} \qquad \text{(E.2)}$$

4.6 Response Spectrum

Example 4.14

Response Spectrum of a Sinusoidal Pulse

Solution

Superimposing the homogeneous solution $x_c(t)$ and the particular solution $x_p(t)$,

$$x(t) = x_c(t) + x_p(t) \qquad \text{(E.3)}$$

$$\mathrm{x(t)} = A\cos\omega_{\mathrm{n}} t + B\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\sin\omega t \qquad \text{(E.4)}$$

$$\text{where} \quad \omega_n = \frac{2\pi}{\tau_n} = \sqrt{\frac{k}{m}} \qquad \text{(E.5)}$$

4.6 Response Spectrum

Example 4.14
Response Spectrum of a Sinusoidal Pulse
Solution

Using the initial conditions, the constants can be found:

$$A = 0, \quad B = -\frac{F_0 \omega}{\omega_n (k - m\omega^2)} \qquad \text{(E.6)}$$

Thus,

$$x(t) = \frac{F_0 / k}{1 - (\omega / \omega_n)^2} \left\{ \sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t \right\}, \qquad 0 \le t \le t_0 \qquad \text{(E.7)}$$

$$\frac{x(t)}{\delta_{st}} = \frac{1}{1 - \left(\dfrac{\tau_n}{2t_0} \right)^2} \left\{ \sin \frac{\pi t}{t_0} - \frac{\tau_n}{2t_0} \sin \frac{2\pi t}{\tau_n} \right\}, \qquad 0 \le t \le t_0 \qquad \text{(E.8)}$$

$$\text{where} \quad \delta_{st} = \frac{F_0}{k} \qquad \text{(E.9)}$$

4.6 Response Spectrum

Example 4.14
Response Spectrum of a Sinusoidal Pulse
Solution

The solution can be expressed as a free vibration solution

$$x(t) = A'\cos\omega_n t + B'\sin\omega_n t, \quad t > t_0 \quad \text{(E.10)}$$

where the constants can be found by:

$$x(t = t_0) = \alpha\left[-\frac{\tau_n}{2t_0}\sin\frac{2\pi t_0}{\tau_n}\right] = A'\cos\omega_n t_0 + B'\sin\omega_n t_0 \qquad \text{(E.11)}$$

$$\dot{x}(t = t_0) = \alpha\left\{\frac{\pi}{t_0} - \frac{\pi}{t_0}\cos\frac{2\pi t_0}{\tau_n}\right\} = -\omega_n A'\sin\omega_n t + \omega_n B'\cos\omega_n t \quad \text{(E.12)}$$

4.6 Response Spectrum

Example 4.14
Response Spectrum of a Sinusoidal Pulse
Solution

Where $\alpha = \dfrac{\delta_{st}}{1-\left(\dfrac{\tau_n}{2t_0}\right)^2}$ (E.13)

Hence, $A' = \dfrac{\alpha\pi}{\omega_n t_0}\sin\omega_n t_0, \qquad B' = -\dfrac{\alpha\pi}{\omega_n t_0}\left[1+\cos\omega_n t_0\right]$ (E.14)

Therefore, $$\frac{x(t)}{\delta_{st}} = \frac{(\tau_n/t_0)}{2\left\{1-(\tau_n/2t_0)^2\right\}}\left[\sin 2\pi\left(\frac{t_0}{\tau_n}-\frac{t}{\tau_n}\right)-\sin 2\pi\frac{t}{\tau_n}\right],$$

$t \geq t_0$ (E.15)

4.6 Response Spectrum

- **Response Spectrum for Base Excitation**

For a harmonic oscillator (an undamped system under free vibration), we notice that

$$\ddot{x}\big|_{\max} = -\omega_n^2 x\big|_{\max} \qquad \text{and} \qquad \dot{x}\big|_{\max} = \omega_n x\big|_{\max}$$

Thus, the acceleration and displacement spectra can be obtained:

$$S_d = \frac{S_v}{\omega_n}, \qquad S_a = \omega_n S_v \qquad (4.38)$$

4.6 Response Spectrum

- **Response Spectrum for Base Excitation**

The velocity response spectrum can be obtained:

$$S_v = \left|\dot{z}(t)\right|_{\max} = \left|\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sqrt{P^2+Q^2}\right|_{\max} \qquad (4.44)$$

Thus the pseudo response spectra are given by:

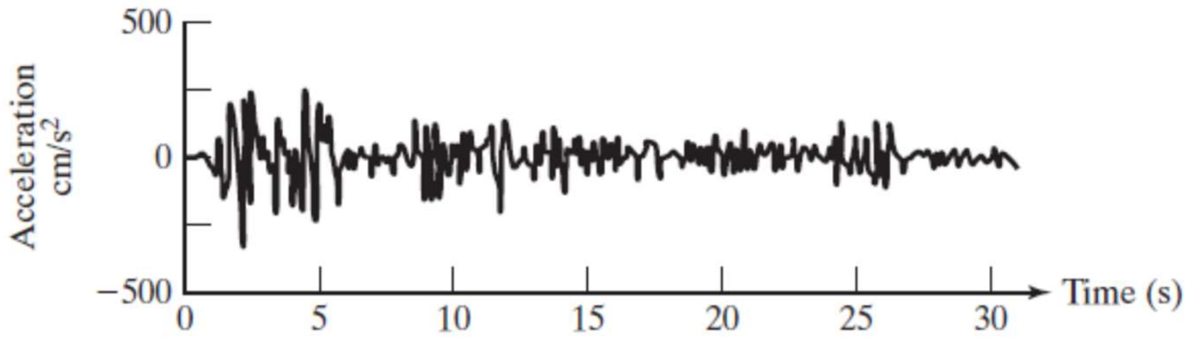
$$S_d = \left|z\right|_{\max} = \frac{S_v}{\omega_n}; \qquad S_v = \left|\dot{z}\right|_{\max}; \qquad S_a = \left|\ddot{z}\right|_{\max} = \omega_n S_v \qquad (4.45)$$

4.6 Response Spectrum

- **Earthquake Response Spectra**

  The most direct description of an earthquake motion in time domain is provided by accelerograms that are recorded by instruments called *strong motion accelerographs*.

  A typical accelerogram is shown in the figure below.



4.6 Response Spectrum

- **Earthquake Response Spectra**

  A response spectrum is used to provide the most descriptive representation of the influence of a given earthquake on a structure of machine. It is possible to plot the maximum response of a single degree freedom system using logarithmic scales.

4.6 Response Spectrum

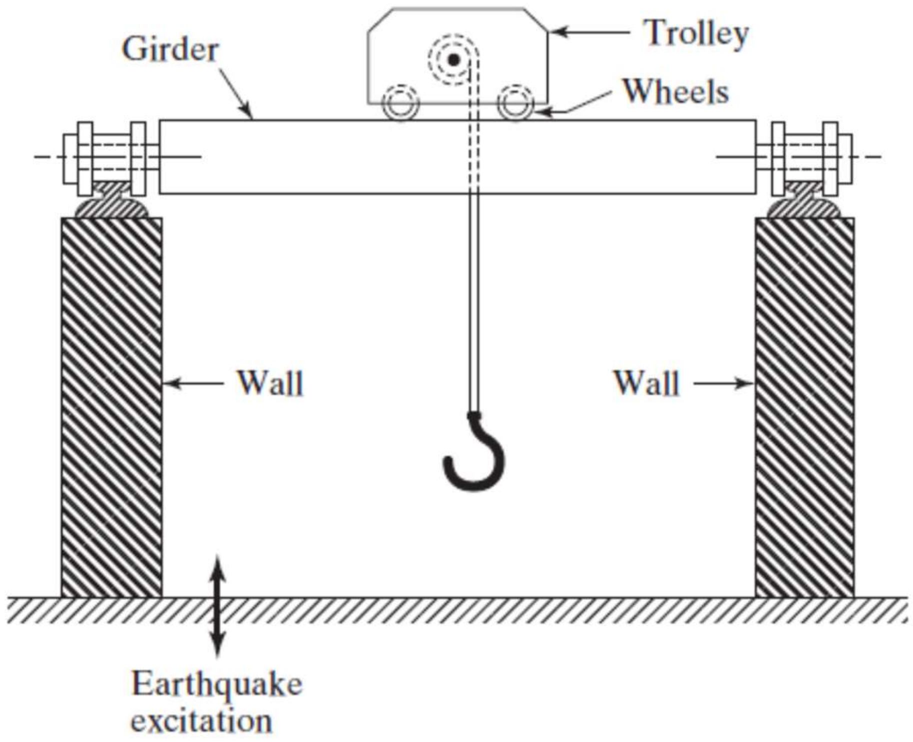
Example 4.17

Derailment of Trolley of a Crane During Earthquake

The trolley of an electric overhead traveling (EOT) crane travels horizontally on the girder as indicated in the figure. Assuming the trolley as a point mass, the crane can be modeled as a single degree of freedom system with a period 2 s and a damping ratio 2%. Determine whether the trolley derails under a vertical earthquake excitation.

4.6 Response Spectrum

Example 4.17
Derailment of Trolley of a Crane During Earthquake



4.6 Response Spectrum

Example 4.17

Derailment of Trolley of a Crane During Earthquake

Solution

For τ_n = 2 s and ζ = 0.02, Fig.4.16 gives the spectral acceleration as Sa = 0.25 g and hence the trolley will not derail.

4.6 Response Spectrum

- **Design Under a Shock Environment**

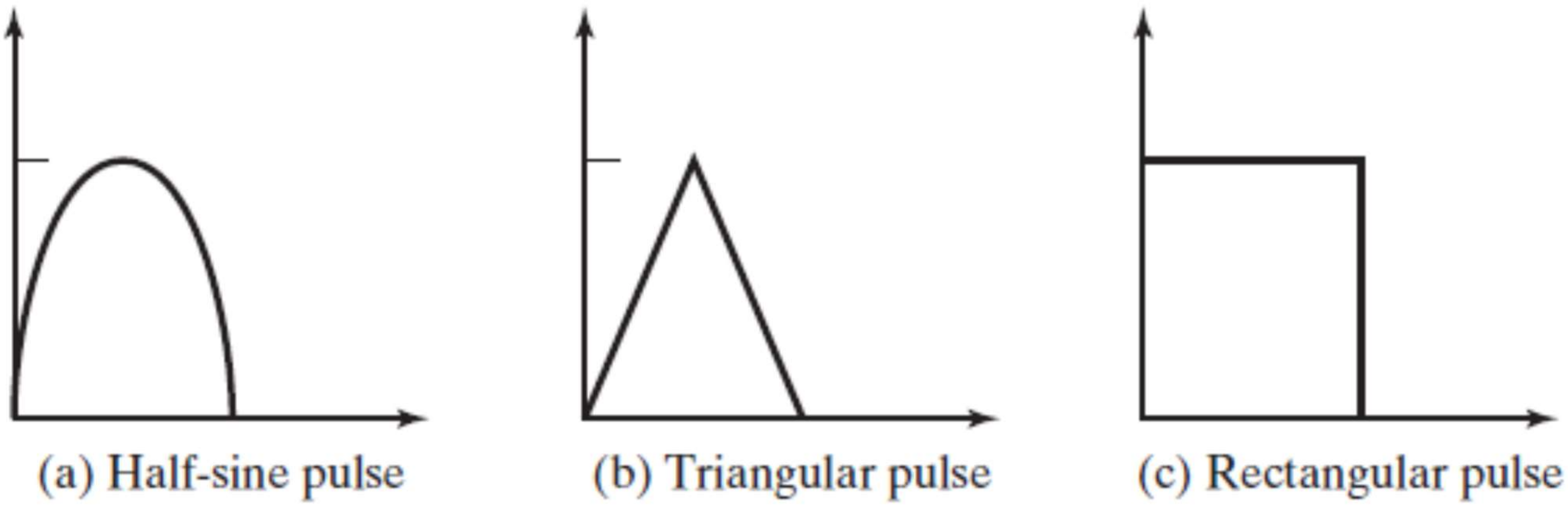
  When a force is applied for short duration, usually for a period of less than one natural time period, it is called a shock load.

  A shock may be described by a pulse shock, velocity shock, or a shock response spectrum. The pulse shocks are introduced by applied forces or displacements in different forms.

  A velocity shock is caused by sudden changes in the velocity. The shock response spectrum describes the way in which a machine or structure responds to a specific shock.

4.6 Response Spectrum

- **Design Under a Shock Environment**



Typical shock pulses

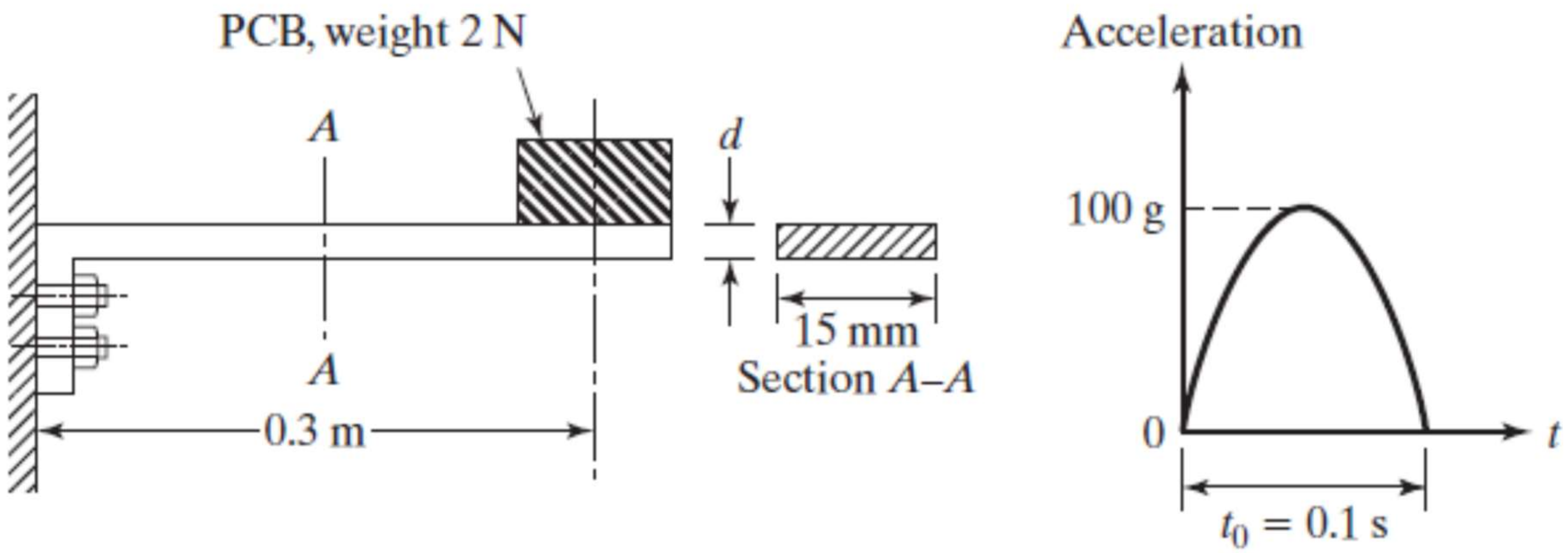
4.6 Response Spectrum

Example 4.18
Design of a Bracket for Shock Loads

A printed circuit board (PCB) is mounted on a cantilevered aluminum bracket, as shown in Figure (a). The bracket is placed in a container that is expected to be dropped from a low-flying helicopter. The resulting shock can be approximated as a half-sine wave pulse, as shown in figure (b). Design the bracket to withstand an acceleration level of 100g under the half-sine wave pulse shown in figure (b). Assume a specific weight of 30 kN/m3, a Young’s modulus of 70 GPa, and a permissible stress of 180 MPa for aluminum.

4.6 Response Spectrum

Example 4.18
Design of a Bracket for Shock Loads



(a) (b)

4.6 Response Spectrum

Example 4.18

Design of a Bracket for Shock Loads

Solution

The self-weight of the beam is given by

$$w = (0.3)(0.015 \times d)(30 \times 10^3) = 135d$$

The total weight is

$$W = \text{Weight of beam} + \text{Weight of PCB} = 135d + 2$$

The area moment of inertia of the cross section of the beam is

$$I = \frac{1}{12} \times 0.015 \times d^3 = 0.00125d^3$$

4.6 Response Spectrum

Example 4.18

Design of a Bracket for Shock Loads

Solution

The static deflection of the beam can be obtained

$$S_{st} = \frac{Wl^3}{3EI} = \frac{(135d + 2)(0.3)^3}{3 \times (70 \times 10^9)(0.00125d^3)} = (1.0286 \times 10^{-10}) \frac{135d + 2}{d^3}$$

We adopt a trial and error procedure to determine the values of unknown.

4.6 Response Spectrum

Example 4.18

Design of a Bracket for Shock Loads

Solution

Assuming d = 10 mm,

$$\delta_{st} = (1.0286 \times 10^{-10}) \frac{135 \times 0.01 + 2}{0.01^3} = 3.446 \times 10^{-4} \text{ m}$$

We have

$$\tau_n = 2\pi \sqrt{\frac{\delta_{st}}{g}} = 2\pi \sqrt{\frac{3.466 \times 10^{-4}}{9.8}} = 0.03726 \text{ s}$$

Hence,

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.03726} = 2.6841$$

4.6 Response Spectrum

Example 4.18

Design of a Bracket for Shock Loads

Solution

The dynamic load acting on the cantilever is given by:

$$P_d = A_a M a_s = (1.1)\left(\frac{3.35}{\text{g}}\right)(100\text{g}) = 368.5\ \text{N}$$

The maximum bending stress at the root of the cantilever bracket can be computed as:

$$\sigma_{\max} = \frac{M_b c}{I} = \frac{(368.5 \times 0.3)\left(\frac{0.01}{2}\right)}{1.25 \times 10^{-9}} = 442.2\ \text{MPa}$$

4.6 Response Spectrum

Example 4.18

Design of a Bracket for Shock Loads

Solution

Since this stress exceeds the permissible value, we assume the next trial value of *d* as 20 mm. This yields:

$$\delta_{st} = (1.0286\times 10^{-10})\frac{135\times 0.02+2}{0.02^3} = 6.0430\times 10^{-5}\ \text{m}$$

$$\tau_{\text{n}} = 2\pi\sqrt{\frac{\delta_{st}}{g}} = 2\pi\sqrt{\frac{6.0430\times 10^{-5}}{9.8}} = 0.01560\,(\text{s})$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.01560} = 6.4$$

4.6 Response Spectrum

Example 4.18
Design of a Bracket for Shock Loads
Solution

The dynamic load can be determined:

$$P_d = A_a M a_s = (1.1)\left(\frac{4.7}{\text{g}}\right)(100\text{g}) = 517\ (\text{N})$$

The maximum bending stress at the root of the bracket will be:

$$\sigma_{\max} = \frac{M_b c}{I} = \frac{(517\times 0.3)\left(\frac{0.02}{2}\right)}{10^{-8}} = 155.1\ \text{MPa}$$

Since this stress is within the permissible limit, the thickness of the bracket can be taken as **20 mm**.

4.7
Laplace Transforms

4.7

4.7 Laplace Transforms

- The Laplace transform of a function x(t) is defined as:

$$x(t=0)\lim_{s\to\infty}[sX(s)] \quad (4.46)$$

- The general solution can be expressed as

$$x(t)=\frac{x_0}{(1-\zeta^2)^{1/2}}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi_1)+\frac{\dot{x}_0}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$+\frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)d\tau$$

4.8

Numerical Methods

4.8

4.8 Numerical Methods

- The determination of the response of a system subjected to arbitrary forcing functions using numerical methods is called numerical simulation.
- Numerical simulations can be used to check the accuracy of analytical solutions, especially if the system is complex.
- Several methods are available for numerically integrating ordinary differential equations.
- The Runge-Kutta methods are quite popular for the numerical solution of differential equations.

4.9

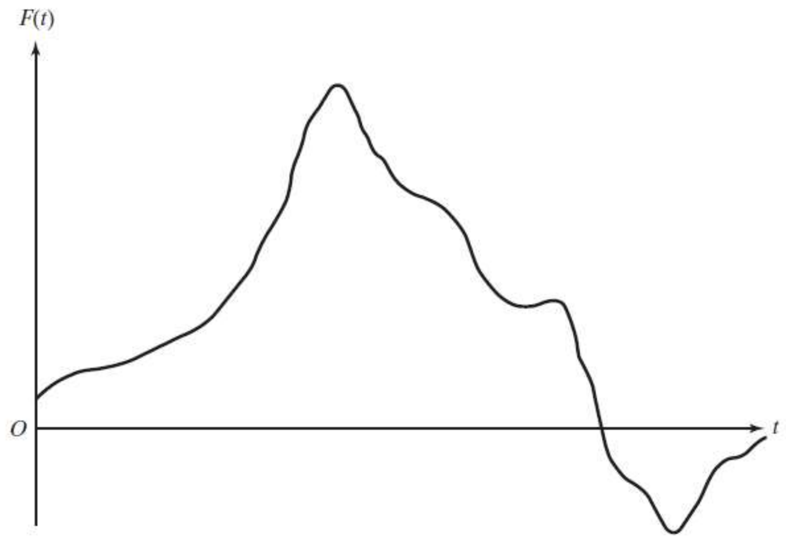
Response to Irregular Forcing Conditions Using Numerical Methods

4.9

4.9 Response to Irregular Forcing Conditions Using Numerical Methods

- Let the function vary with time in an arbitrary manner. The response of the system can be found:

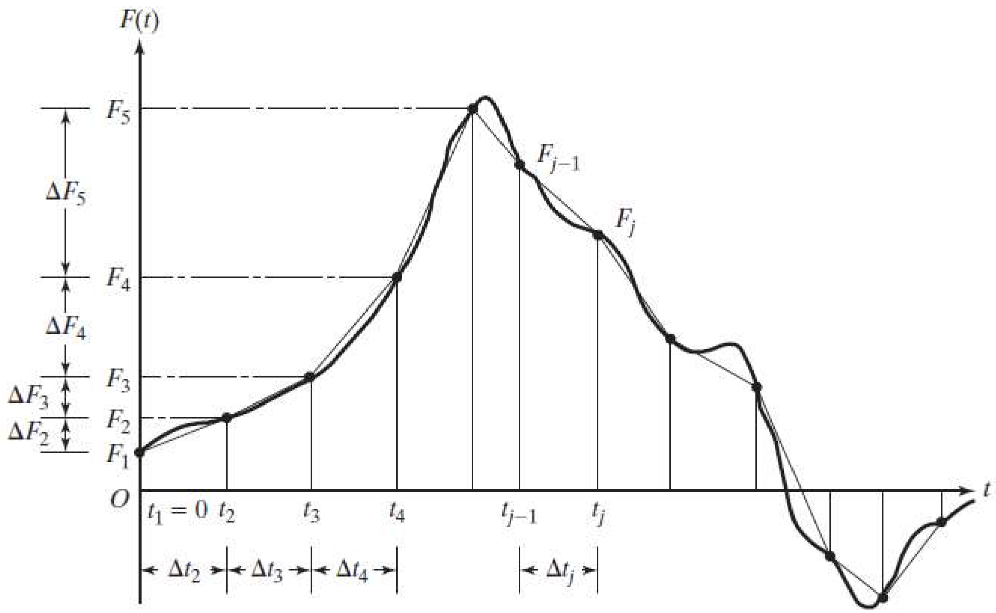
$$x(t) = \frac{1}{k}\sum_{i=1}^{j-1} \Delta F_i \left[1 - e^{-\zeta\omega_n (t-t_i)} \times \left\{ \cos\omega_d (t-t_i) + \frac{\zeta\omega_n}{\omega_d} \sin\omega_d (t-t_i) \right\} \right]$$



4.9 Response to Irregular Forcing Conditions Using Numerical Methods

- Thus the response of the system at $t = t_j$ becomes

$$x_j = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\left[1 - e^{-\zeta\omega_n(t_j - t_i)} \times \left\{\cos\omega_d(t_j - t_i) + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d(t_j - t_i)\right\}\right]$$



4.9 Response to Irregular Forcing Conditions Using Numerical Methods

Example 4.31

Damped Response Using Numerical Methods

Find the response of a spring-mass-damper system subjected to the forcing function

$$F(t) = F_0\left(1 - \sin\frac{\pi t}{2t_0}\right) \qquad \text{(E.1)}$$

in the interval $0 \le t \le t_0$, using a numerical procedure.

Assume F_0=1, k=1, m=1, ζ=0.1, and t0=$\tau_n/2$, where τ_n denotes the natural period of vibration given by

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(k/m)^{1/2}} = 2\pi \qquad \text{(E.2)}$$

The values of x and $\dot{x}$ at t=0 are zero

4.9 Response to Irregular Forcing Conditions Using Numerical Methods

Example 4.31

Damped Response Using Numerical Methods

Solution

For the numerical computations, the time interval 0 to t_0 is divided into 10 equal steps with

$$\Delta t_i = \frac{t_0}{10} = \frac{\pi}{10}; \qquad i = 2,3,\ldots,11 \qquad \text{(E.3)}$$

4 different methods are used to approximate the forcing function *F(t)*.

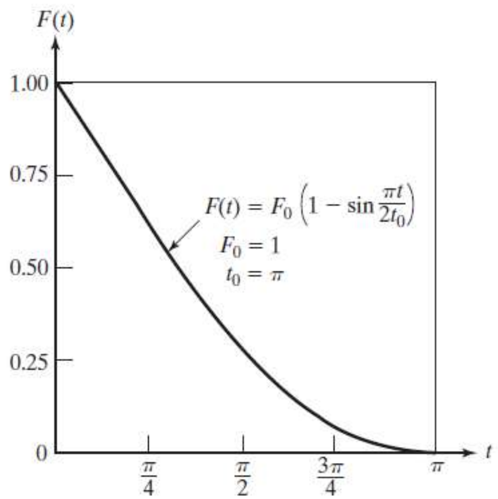
4.9 Response to Irregular Forcing Conditions Using Numerical Methods

Example 4.31

Damped Response Using Numerical Methods

Solution

In the figure, *F(t)* is approximated by a series of rectangular impulses, each starting at the beginning of the corresponding time step.



4.9 Response to Irregular Forcing Conditions Using Numerical Methods

Example 4.31

Damped Response Using Numerical Methods

Solution

In Fig. 4.36, piecewise linear (trapezoidal) impulses are used to approximate the forcing function $F(t)$. The numerical results are given in Table 4.2. The results can be improved by using a higher-order polynomial for interpolation instead of the linear function.

4.9 Response to Irregular Forcing Conditions Using Numerical Methods

Example 4.31

Damped Response Using Numerical Methods

Solution

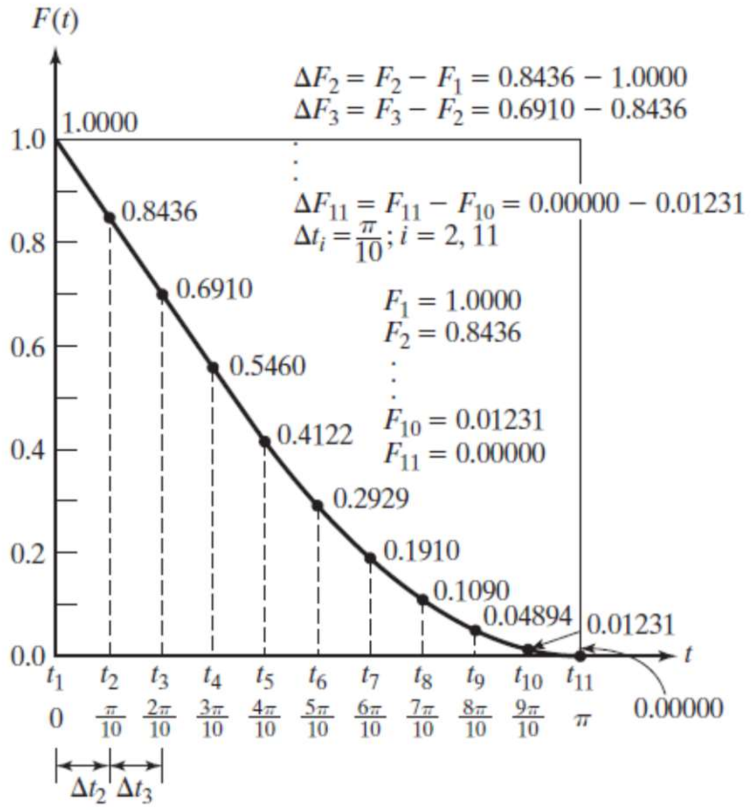


TABLE 4.2 Response of the system

i	t_i	$x(t_i)$ Obtained According to Fig. 4.36 (Idealization 4)
1	0	0.00000
2	0.1π	0.04541
3	0.2π	0.16377
4	0.3π	0.32499
5	0.4π	0.49746
6	0.5π	0.65151
7	0.6π	0.76238
8	0.7π	0.81255
9	0.8π	0.79323
10	0.9π	0.70482
11	π	0.55647

