高瞻計畫_振動學課程
Lecture 2: Single Degree of Freedom Vibration (I)

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Outline

- Introduction
- Mass-spring system modeling
- Energy method
- Mass-spring-damper systems
- 2^{nd} order ODE: a brief catch up
- Responses
- Simple examples
- Case studies

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Part 1. Introduction

Procedure of Solving an Engineering Vibration Problem (I)

- Engineering problem statement
  - Identify problems or performance index
    - Stress? Fatigue life? Vibration amplitude?
    - Bandwidth? Sensitivity?
- Modeling engineering problems
  - Engineering problem → engineering models → vibration models
- Analyzing vibration models
  - Stiffness, natural frequency, ….

Procedure of Solving an Engineering Vibration Problem (II)

- Solving vibration problems
  - Mathematical techniques
    - ODE, Linear algebra, PDE, Fourier series
    - Finite element analysis
- Interpretation and Prediction
  - Convert mathematical results to engineering design
  - Vibration solution → engineering solution
- Experimental validation
  - Verify and test the results
  - Modify the design

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Vibration Modeling of Engineering Systems (I)

- Physical systems
  - Identify the key issue for the physical systems or problems
  - E.g. Extra noise; in sufficient bandwidth; fatigue damage
- convert the physical system to an engineering model
  - By neglecting unnecessary details
- Reduce the engineering model to a vibration model

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Vibration Modeling of Engineering Systems (II)

- Vibration model is not unique
  - Depends on what you want
  - Simple / efficient vs. complicate / detail
- Single degree of freedom (SDOF) model
  - Simple, efficient, but lack of detail
- Multiple degree of freedom (MDOF) model
  - w/ technical detail but need mathematical effort
- Continuous model
  - More realistic physical details
  - Much more complicate expression

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Analyzing the Vibration Model

- Obtaining equations of motion from vibration models
- Fundamental physical laws
  - Newton's 2nd law
  - Hamilton's principle
- Analyzing techniques
  - Free body diagrams
  - Calculus of variation

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Solutions of the Vibration Model

- To solve the equations of motion
  - Equations of motion
  - Appropriate boundary conditions
  - Given initial conditions
- Mathematical techniques
  - Ordinary differential equations (ODE)
  - Linear algebra
  - Partial differential equations (PDE)
  - Fourier analysis

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Prediction Based on the Solution

- Analytical solutions
  - Provide performance correlations with design parameters
- Design optimization with constraints

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Experimental Validation

- Vibration testing:
  - To find problems
    - E.g., excessive noise, resonance
  - To find necessary physical parameters
    - E.g., system damping, system spring constants
  - To validate the analysis results
    - Modification of analysis model

Design Recommendation

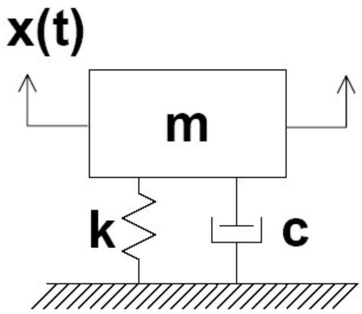
- Vibration analysis
  - Provide strategies for solving engineering problems with vibration concerns
  - Provide solutions for improving performances of engineering products
- Vibration testing
  - Identify the sources of engineering problems
  - Obtain model parameters for vibration analysis
  - Validate vibration analysis results

Part II. Mass-Spring System Modeling

SDOF Definitions

Assumptions

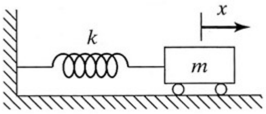
- *lumped mass*
- *stiffness proportional to displacement*
- *damping proportional to velocity*
- *linear time invariant*
- *2nd order differential equations*



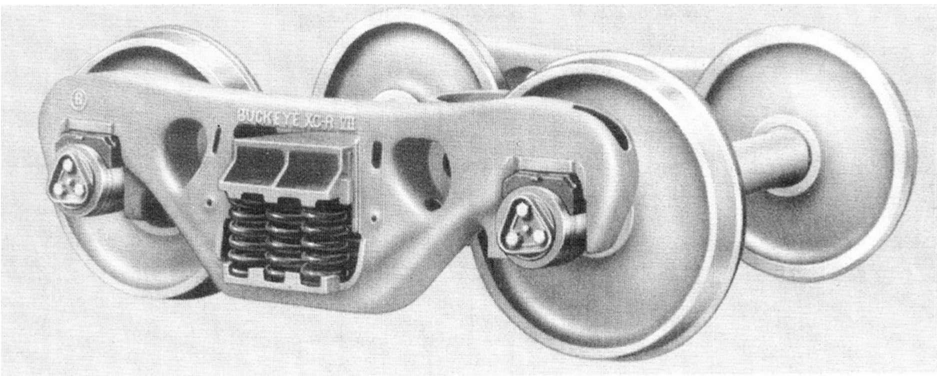
SDOF Systems

- Many engineering problems can be simplified into SDOF models
  - E.g., tall building; suspension systems

Example of SDOF Systems

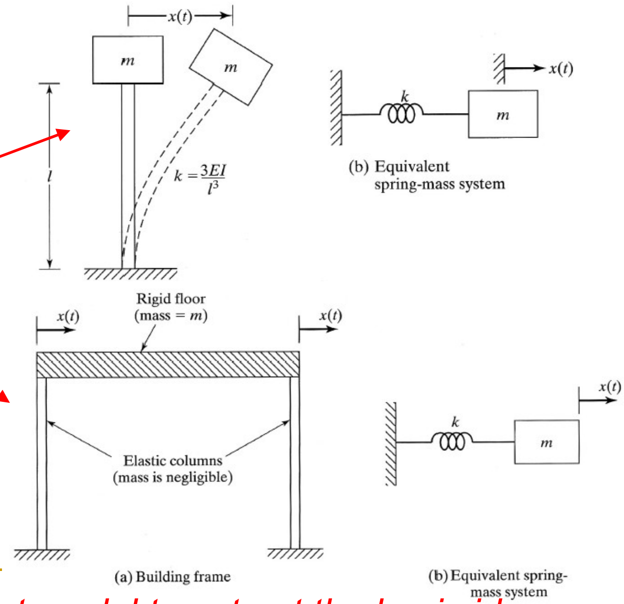


(b) Spring-mass system



Importance of SDOF Model

FIGURE 2.3 The space needle (structure).



- *Provide a simplest model to extract the basic idea*

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Importance of SDOF Model (Cont'd)

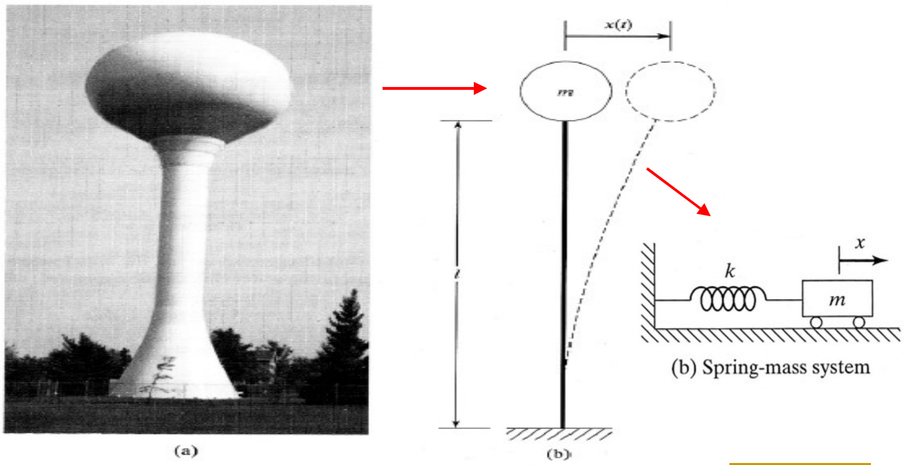

FIGURE 2.10 Elevated tank. (Photo courtesy of West Lafayette Water Company.)

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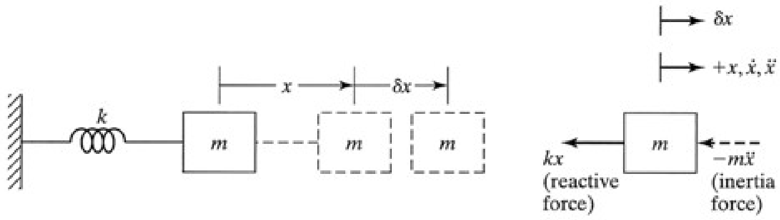
Mathematical Modeling (I)

- Starting from Newton's 2nd law:

$$\vec{F}(t) = m\frac{d^2\vec{x}(t)}{dt^2} = m\ddot{\vec{x}}$$

- By free-body diagram

$$-kx - m\ddot{x} = 0 \quad \text{or} \quad m\ddot{x} + kx = 0$$



(a) Mass under a displacement x

(b) Free-body diagram

Mathematical Modeling (II)

- Alternative standard form

$$\ddot{x} + \frac{k}{m}x = 0 \quad \text{or} \quad \ddot{x} + \omega_n^2 x = 0$$

- Natural frequency

$$\omega_n = \sqrt{\frac{k}{m}}$$

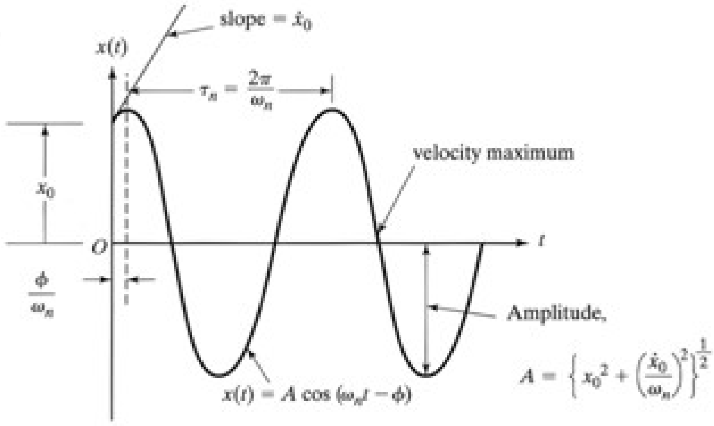
- Typical response

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

$$x(t = 0) = A_1 = x_0$$
$$\dot{x}(t = 0) = \omega_n A_2 = \dot{x}_0$$

Mathematical Modeling (III)

- Typical response



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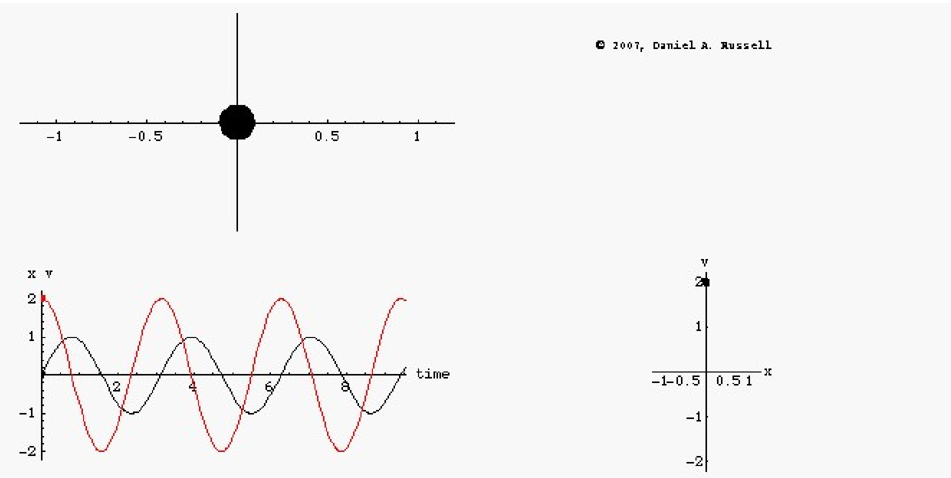
Phase Diagram: Undamp System



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Spring Constants for Typical Engineering Structures

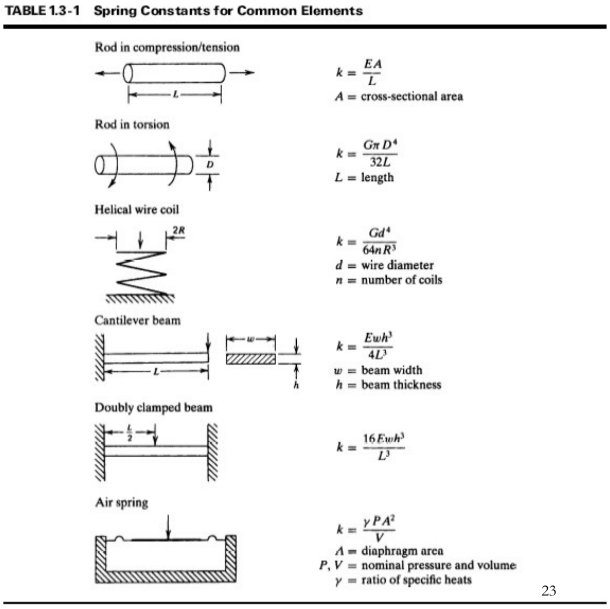

TABLE 1.3-1 Spring Constants for Common Elements

Rod in compression/tension	$k = \frac{EA}{L}$; A = cross-sectional area
Rod in torsion	$k = \frac{G\pi D^4}{32L}$; L = length
Helical wire coil	$k = \frac{Gd^4}{64nR^3}$; d = wire diameter; n = number of coils
Cantilever beam	$k = \frac{Ewh^3}{4L^3}$; w = beam width; h = beam thickness
Doubly clamped beam	$k = \frac{16Ewh^3}{L^3}$
Air spring	$k = \frac{\gamma PA^2}{V}$; A = diaphragm area; P, V = nominal pressure and volume; γ = ratio of specific heats

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Calculating *RMS*

A = peak value

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt = \text{average value}$$

$$\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt = \text{mean - square value}$$

$$x_{rms} = \sqrt{\bar{x}^2} = \text{root mean square value}$$

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The Decibel or dB scale

It is often useful to use a logarithmic scale to plot vibration levels (or noise levels). One such scale is called the *decibel* or dB scale. The dB scale is always relative to some reference value x_0. It is define as:

$$dB = 10\log_{10}\left(\frac{x}{x_0}\right)^2 = 20\log_{10}\left(\frac{x}{x_0}\right)$$

For example: if an acceleration value was 19.6m/s^2 then relative to 1*g* (or 9.8m/s^2) the level would be 6dB,

$$10\log_{10}\left(\frac{19.6}{9.8}\right)^2 = 20\log_{10}(2) = 6dB$$

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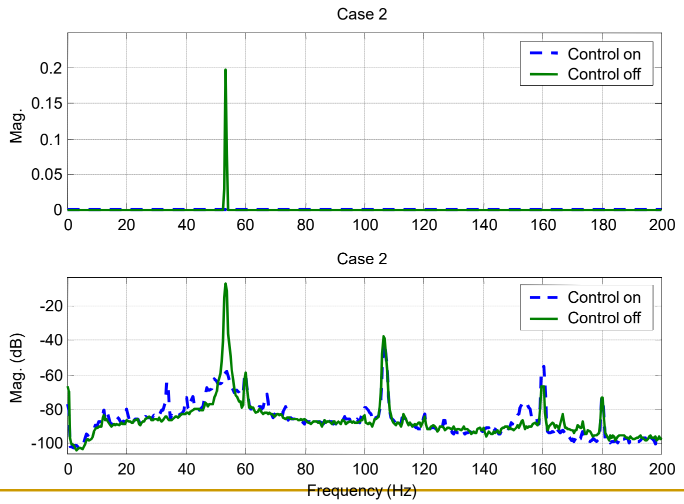
Comparison of Linear and dB plots



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Part III. Energy Methods

Introduction

- An alternative approach for modeling SDOF systems
- Can obtain approximate SDOF model for engineering systems
  - Assume shape
  - Energy conservation
  - Rayleigh or Rayleigh-Ritz methods
  - Eventually, finite element method

Energy Approach for Exact Mass-Spring Systems

- Energy conservation
  - T+V = constant

$$\frac{d}{dt}(T + U) = 0$$

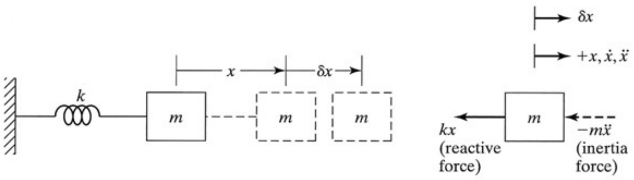
$$T = \tfrac{1}{2}m\dot{x}^2$$

$$U = \tfrac{1}{2}kx^2$$

$$m\ddot{x} + kx = 0$$

$$W = mg = k\delta_{st}$$

$$m\ddot{x} + kx = 0$$



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Insights: Rayleigh Method

- In a no-loss system, the mechanical energy is conservative
  - Maximum kinetic energy = Maximum potential energy
  - Mainly used for calculating approximated natural frequency
  - Need an reasonable assumed mode shape

$$T_1 + U_1 = T_2 + U_2$$

$$T_1 + 0 = 0 + U_2$$

$$T_{max} = U_{max}$$

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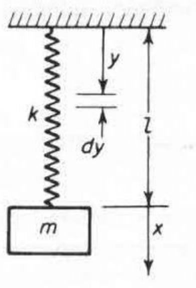
2.5 Rayleigh's Energy Method

EXAMPLE 2.8

Effect of Mass on ω_n of a Spring



method to determine the natural frequency. Let l be the total length of the spring. If x denotes the displacement of the lower end of the spring (or mass m), the displacement at distance y from the support is given by $y(x/l)$. Similarly, if $\dot{x}$ denotes the velocity of the mass m, the velocity of a spring element located at distance y from the support is given by $y(\dot{x}/l)$. The kinetic energy of the spring element of length dy is

$$dT_s = \frac{1}{2}\left(\frac{m_s}{l}dy\right)\left(\frac{y\dot{x}}{l}\right)^2 \tag{E.1}$$

where m_s is the mass of the spring. The total kinetic energy of the system can be expressed as

$$\begin{aligned} T &= \text{kinetic energy of mass } (T_m) + \text{kinetic energy of spring } (T_s) \\ &= \frac{1}{2}m\dot{x}^2 + \int_{y=0}^{l} \frac{1}{2}\left(\frac{m_s}{l}dy\right)\left(\frac{y^2\dot{x}^2}{l^2}\right) \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2 \end{aligned} \tag{E.2}$$

The total potential energy of the system is given by

$$U = \tfrac{1}{2}kx^2 \tag{E.3}$$

By assuming a harmonic motion

$$x(t) = X\cos\omega_n t \tag{E.4}$$

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2.5 Rayleigh's Energy Method

EXAMPLE 2.8

Effect of Mass on ω_n of a Spring

where X is the maximum displacement of the mass and ω_n is the natural frequency, the maximum kinetic and potential energies can be expressed as

$$T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)X^2\omega_n^2 \tag{E.5}$$

$$U_{\max} = \frac{1}{2}kX^2 \tag{E.6}$$

By equating $T_{\max}$ and $U_{\max}$, we obtain the expression for the natural frequency:

$$\omega_n = \left(\frac{k}{m + \frac{m_s}{3}}\right)^{1/2} \tag{E.7}$$

Thus the effect of the mass of the spring can be accounted for by adding one-third of its mass to the main mass [2.3].

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Part IV: Mass-Spring-Damper Systems

Introduction

- Add damper for consideration
- Simplified damper (Linear damper)
- Can use Newton's 2^{nd} law or Lagrange's equation for modeling

Linear Viscous Dampers

- Force is assumed to be proportional to the relative velocity, i.e.,

$$F = -c\dot{x}$$

  - This is true for viscous oil at low velocity
- In practical, other important forms
  - Aerodynamic drag

$$F = -c\dot{x}^2$$

  - Coulomb friction

$$F = -\operatorname{sgn}\dot{x}$$

- Non of them can complete discribe damping behavior

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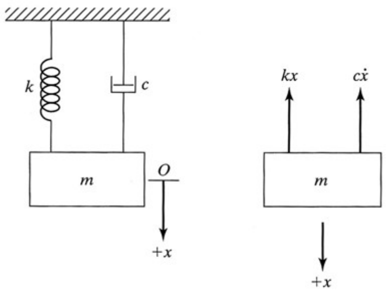
Mathematical Modeling

- By free body diagram, the equation can be obtained as

$$m\ddot{x} + c\dot{x} + kx = 0$$



System Free-body diagram

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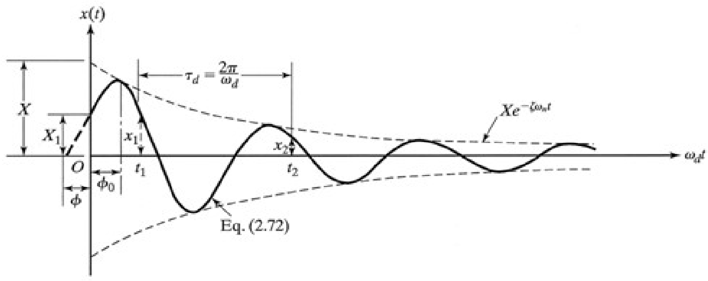
Mathematical Modeling (II)

- Typical solution form

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

- Typical response



Part V: Brief Capture of 2nd Order ODEs and Vibration Responses

- 2nd order ODE plays important role in vibration of SDOF system

Generic Equation and Solution Forms

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

- Can be use to describe the behavior of many engineering systems
  - Mechanical vibration
  - Oscillatory circuits
  - Hydraulic actuator,.. Etc
- Understand the solution form is important for further understanding the vibration insight
- Analogous system

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Lack of Damping Term (c=0)

$$x(t) = Ce^{st}$$

$$ms^2 + k = 0$$

$$s = \pm\left(-\frac{k}{m}\right)^{1/2} = \pm i\omega_n$$

$$\omega_n = \left(\frac{k}{m}\right)^{1/2}$$

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$$

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

$$x(t = 0) = A_1 = x_0$$
$$\dot{x}(t = 0) = \omega_n A_2 = \dot{x}_0$$

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

$$A_1 = A \cos \phi$$
$$A_2 = A \sin \phi$$

$$A = (A_1^2 + A_2^2)^{1/2} = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2} = \text{amplitude}$$

$$\phi = \tan^{-1}\left(\frac{A_2}{A_1}\right) = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right) = \text{phase angle}$$

$$x(t) = A \cos (\omega_n t - \phi)$$

$$A_1 = A_0 \sin \phi_0$$
$$A_2 = A_0 \cos \phi_0$$

$$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

$$A_0 = A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2}$$

$$\phi_0 = \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right)$$

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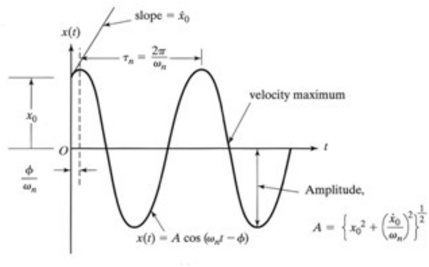
Response

- To solve SDOF vibration problem, one needs to provide two initial conditions
  - Initial position
  - Initial velocity



- Problem can be solved by either
  - Analytical or
  - Runge-Kutta (for complicated nonlinear ODE or MDOF ODEs)

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With Damping Term

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x(t) = Ce^{st}$$

$$ms^2 + cs + k = 0$$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t}$$

- Over damped situation ($c^2>4mk$)
- Critical damped situation ($c^2=4mk$)
- Under damped situation ($c^2<4mk$)

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Damping Ratio

- Critical damping coefficient c_c
  - The damping coefficient for resulting fastest settling

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

- Damping ratio ζ
  - The ratio between damping coefficient and c_c

$$\zeta = c/c_c$$

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta\omega_n$$

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Solutions

- Over damped solution

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$
$$= C_1 e^{\left\{-\frac{c}{2m}+\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m}-\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\right\}t}$$

- Under damped solution

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

- Critical damped solution

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t}$$

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Possibility 1. Critically damped motion

Critical damping occurs when ζ=1. The damping coefficient *c* in this case is given by:

$$\zeta = 1 \Rightarrow c = \underbrace{c_{cr} = 2\sqrt{km}}_{\text{definition of critical damping coefficient}} = 2m\omega_n$$

Solving for λ then gives,

A repeated, real root

The solution then takes the form

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t}$$

Needs two independent solutions, hence the *t* in the second term

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Critically damped motion

a_1 and a_2 can be calculated from initial conditions *(t=0)*,

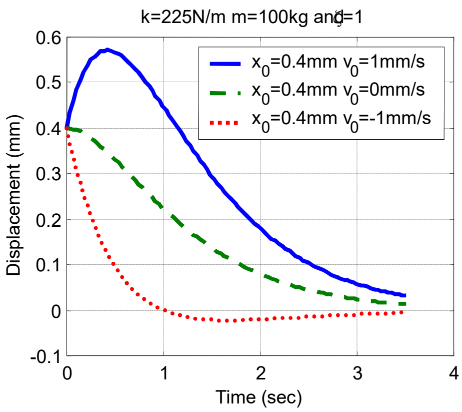
$$x = (a_1 + a_2 t)e^{-\omega_n t}$$

$$\Rightarrow a_1 = x_0$$

$$v = (-\omega_n a_1 - \omega_n a_2 t + a_2)e^{-\omega_n t}$$

$$v_0 = -\omega_n a_1 + a_2$$

$$\Rightarrow a_2 = v_0 + \omega_n x_0$$



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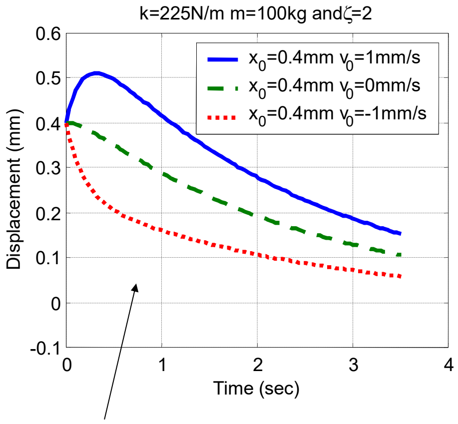
Possibility 2: Overdamped motion

An overdamped case occurs when ζ>1. Both of the roots of the equation are again real.

a₁ and *a₂* can again be calculated from initial conditions *(t=0)*,

$$a_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

$$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$



Slower to respond than critically damped case

Possibility 3: Underdamped motion

An underdamped case occurs when ζ<1. The roots of the equation are complex conjugate pairs. This is the most common case and the only one that yields oscillation.

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n j\sqrt{1-\zeta^2}$$

$$x(t) = e^{-\zeta\omega_n t}(a_1 e^{j\omega_n t\sqrt{1-\zeta^2}} + a_2 e^{-j\omega_n t\sqrt{1-\zeta^2}})$$

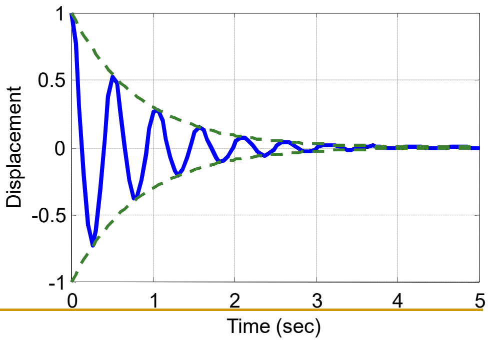
$$= Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

The frequency of oscillation ω_d is called the *damped natural frequency* is given by.

Underdamped motion

A and ϕ can be calculated from initial conditions *(t=0)*,

- Gives an oscillating response with exponential decay
- Most natural systems vibrate with and underdamped response
- See Window 1.5 for details and other representations



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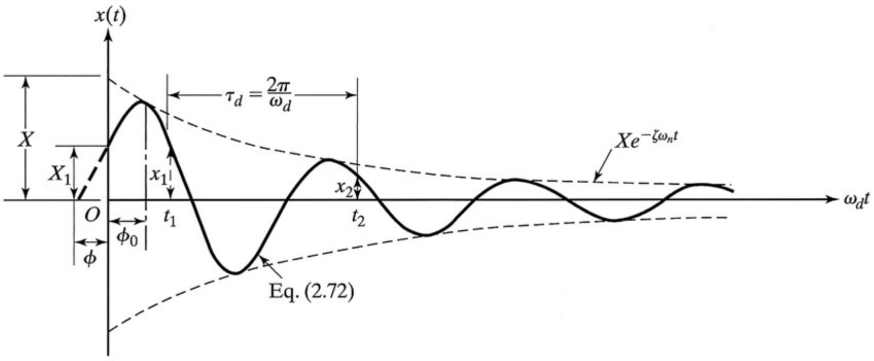
Under Damped Response

$$\omega_d = \omega_n \sqrt{1-\varsigma^2}$$ damped natural frequency



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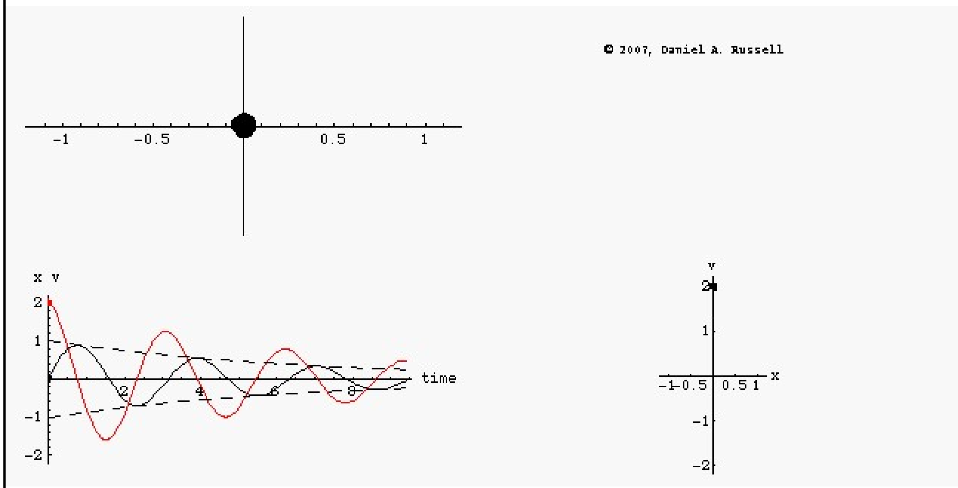
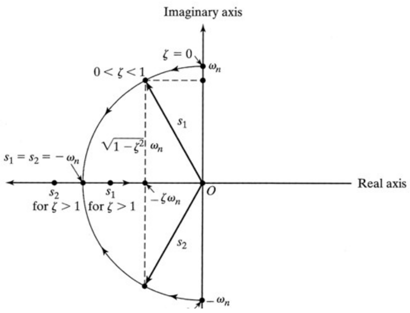
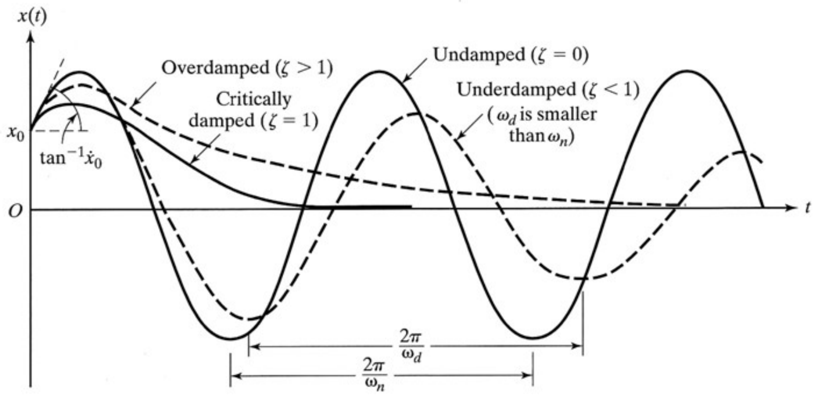
Phase Diagram: Damped System



Damped Responses



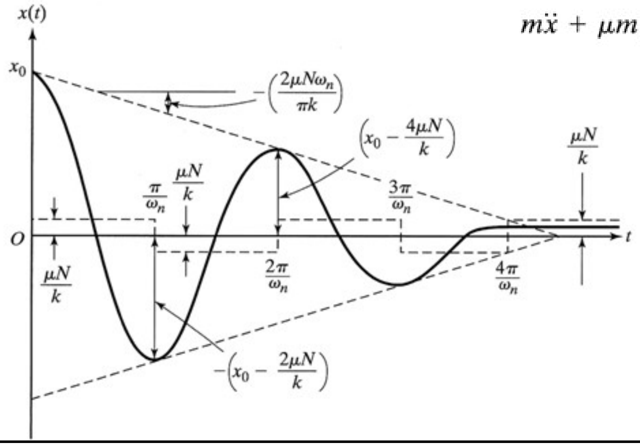


Coulomb (Friction) Damping

$$-kx + \mu N = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = \mu N$$

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu N}{k}$$

$$m\ddot{x} + \mu m\, g\, \mathrm{sgn}(\dot{x}) + kx = 0$$



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Part VI: Simple Examples

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Problem 1: Harmonic Response of Water Tank (Rao 2.1)

After p. 26

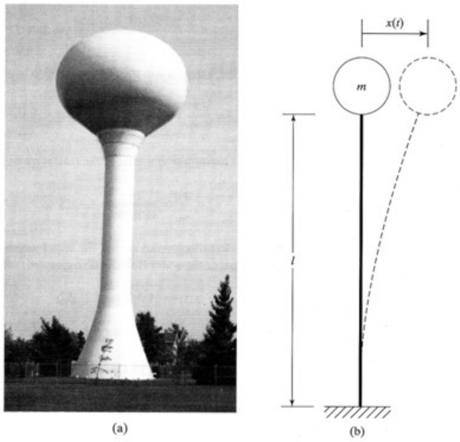
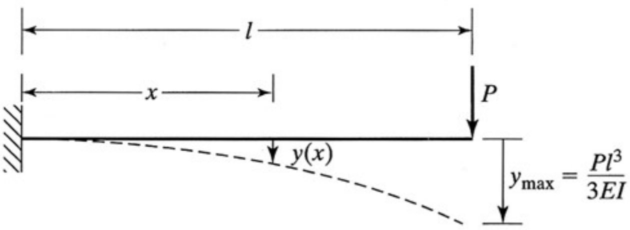


FIGURE 2.10 Elevated tank. (Photo courtesy of West Lafayette Water Company.)

The column of the water tank shown in Fig. 2.10(a) is 90 m high and is made of reinforced concrete with a tubular cross section of inner diameter 2.4 m and outer diameter 3 m. The tank has a mass of 3×10^5 kg with water. By neglecting the mass of the column and assuming the Young's modulus of reinforced concrete as 30 GPa, determine the following:

Problem 2: Effect of Beam Mass on Natural Frequency of Water Tank (Rao 2.9)



Find the natural frequency of transverse vibration of the water tank considered in Example 2.1 and Fig. 2.10 by including the mass of the column.

After p32

Problem 3: Response of Anvil of Forging Hammer (Rao 2.10)

After p52

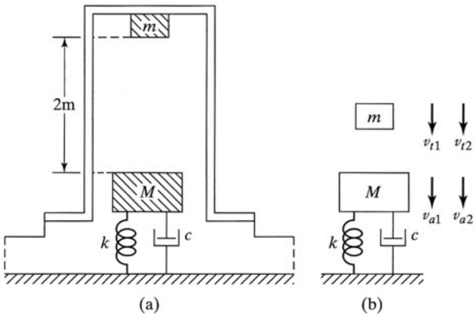


FIGURE 2.30 Forging hammer.

The anvil of a forging hammer weighs 5,000 N and is mounted on a foundation that has a stiffness of 5×10^6 N/m and a viscous damping constant of 10,000 N-s/m. During a particular forging operation, the tup (i.e., the falling weight or the hammer) weighing 1,000 N, is made to fall from a height of 2 m on to the anvil (Fig. 2.30a). If the anvil is at rest before impact by the tup, determine the response of the anvil after the impact. Assume that the coefficient of restitution between the anvil and the tup is 0.4.

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Problem 4. Shock Absorber of a Motorcycle (Rao 2.11)

After p52

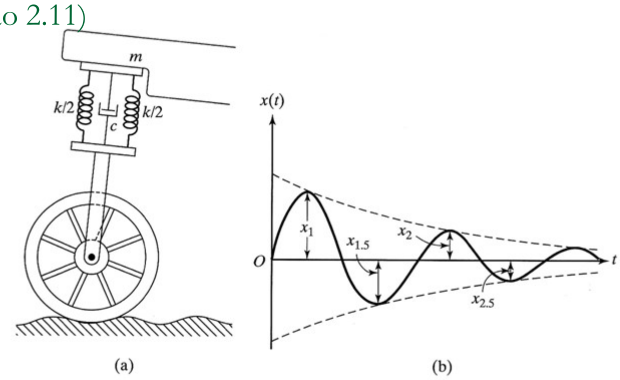

FIGURE 2.31 Shock absorber of a motorcycle.

An underdamped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig. 2.31a). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig. 2.31(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 s and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

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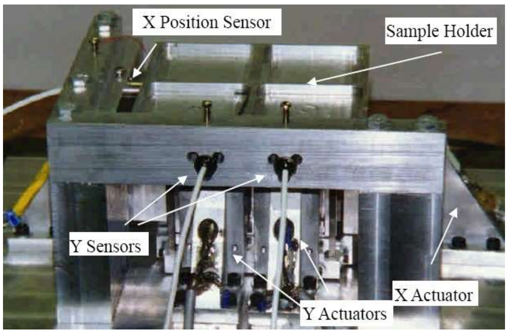
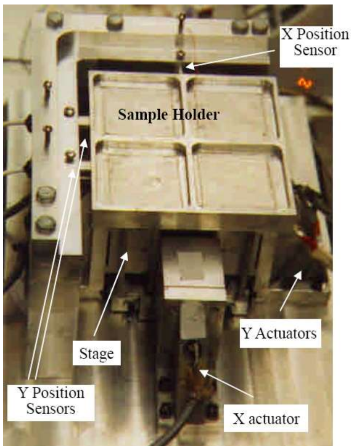
Part VII: Case Studies

Case I: Precision Stage Modeling (MIT)





- 3 DoF precision stage for precision metrology

Modeling

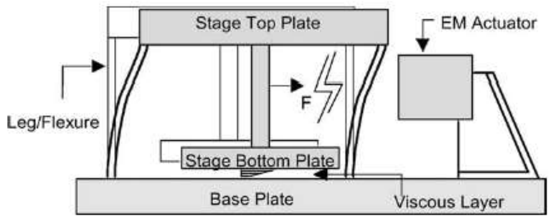
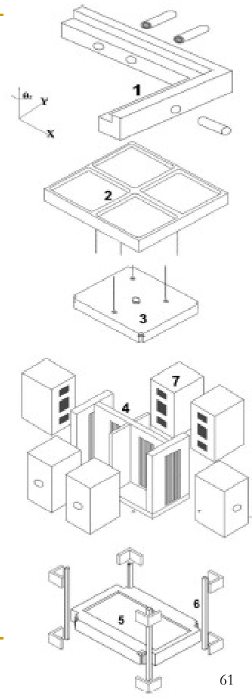


Fig. 4. Schematic plot of stage motion.

$$K_{XY} = \frac{48EI}{L^3}, \qquad f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m_e}},$$

$$K_\theta \equiv \frac{T}{\theta} = \frac{4 \times 0.1406G(2a)^4}{\beta L} + \frac{48EId^2}{L^3},$$



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