Mechanical Vibrations

Chapter 6

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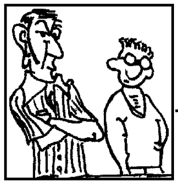


MDOF Equations of Motion

Equation of Motion for 2 DOF

$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

can be written in compact matrix form as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\}$$

MDOF Equations of Motion

This coupled set of equations can be uncoupled by performing an eigensolution to obtain 'eigenpairs' for each mode of the system, that is

'eigenvalues' and 'eigenvectors'

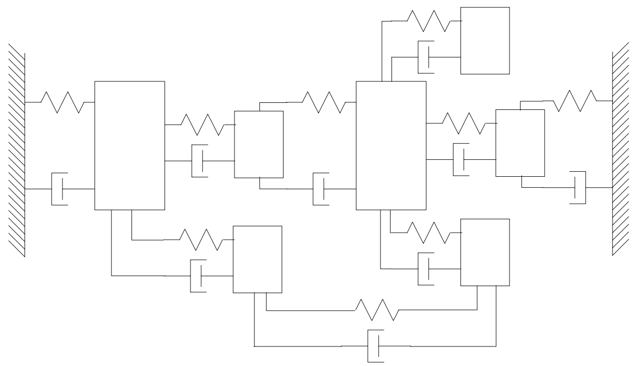
or

'frequencies' (poles) and 'mode shapes'

MDOF Equations of Motion

Equation of Motion

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$$



Eigensolution

$$[[K]-\lambda[M]]\{x\}=0$$

Frequencies (eigenvalues) and Mode Shapes (eigenvectors)

$$\begin{bmatrix} \diagdown & & \\ & \Omega^2 & \\ & & \diagdown \end{bmatrix} = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \diagdown \end{bmatrix} \text{ and } [U]=[\{u_1\} \quad \{u_2\} \quad \cdots]$$

MDOF - Orthogonality of Eigenvectors

Normal modes are 'orthogonal' with respect to the system mass and stiffness matrices.

The eigen problem can be written as

$$[K][U] = [M][U]\left\lceil \Omega^2 \right\rfloor$$

For the ith mode of the system, the eigenproblem can be written as

$$[K]\{u_i\} = \lambda_i [M]\{u_i\} \qquad (6.6.1)$$

MDOF - Orthogonality of Eigenvectors

Premultiply that equation by the transpose of a different vector u_j

$$\{u_j\}^T[K]\{u_i\} = \lambda_i\{u_j\}^T[M]\{u_i\} \qquad (6.6.2)$$

and write a second equation for u_j and premultiply it by the transpose of vector u_i

$$\{u_i\}^T[K]\{u_j\} = \lambda_j\{u_i\}^T[M]\{u_j\} \qquad (6.6.3)$$

MDOF - Orthogonality of Eigenvectors

Subtracting 6.6.3 from 6.6.2, yields

$$\left(\lambda_i - \lambda_j\right)\{u_i\}^T[M]\{u_j\} = 0 \qquad (6.6.2)$$

Since the eigenvalues for each mode are different,

$$\{u_i\}^T[M]\{u_j\} = 0 \qquad i \neq j \qquad (6.6.6)$$

When i=j, then the values are not zero

$$\left.\begin{array}{r} \{u_i\}^T[M]\{u_i\} = \overline{m}_{ii} \\ \{u_i\}^T[K]\{u_i\} = \overline{k}_{ii} \end{array}\right\} \quad i = j \qquad (6.6.8)$$

Modal Matrix and Modal Space Transformation

Define the modal matrix as the collection of modal vectors for each mode organized in column fashion in the modal matrix

$$[U] = [\{u_1\} \quad \{u_2\} \quad \{u_3\} \quad \cdots]$$

This modal matrix is then used to define the modal transformation equation with a new coordinate with 'p' as the principal coordinate

$$\{x\} = [U]\{p\} = [\{u_1\} \quad \{u_2\} \quad \cdots]\begin{Bmatrix} p_1 \\ p_2 \\ \vdots \end{Bmatrix}$$

Modal Space Transformation

Substitute the modal transformation into the equation of motion

$$[M][U]\{\ddot{p}\} + [C][U]\{\dot{p}\} + [K][U]\{p\} = \{F\}$$

In order to put the equations in normal form, this equation must be premultiplied by the transpose of the projection operator to give

$$[U]^T[M][U]\{\ddot{p}\} + [U]^T[C][U]\{\dot{p}\} + [U]^T[K][U]\{p\} = [U]^T\{F\}$$

Modal Space Transformation

The first term of the modal acceleration can be expanded as

$$[U]^T[M][U] = \begin{bmatrix} \left(\{u_1\}^T[M]\{u_1\}\right) & \left(\{u_1\}^T[M]\{u_2\}\right) & \left(\{u_1\}^T[M]\{u_3\}\right) & \cdots \\ \left(\{u_2\}^T[M]\{u_1\}\right) & \left(\{u_2\}^T[M]\{u_2\}\right) & \left(\{u_2\}^T[M]\{u_3\}\right) & \cdots \\ \left(\{u_3\}^T[M]\{u_1\}\right) & \left(\{u_3\}^T[M]\{u2\}\right) & \left(\{u_3\}^T[M]\{u_3\}\right) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Modal Space Transformation

Recall from orthogonality that

$$\{u_i\}^T[M]\{u_j\} = 0 \qquad i \neq j$$

so that

$$[U]^T[M][U] = \begin{bmatrix} \left(\{u_1\}^T[M]\{u_1\}\right) & 0 & 0 & \cdots \\ 0 & \left(\{u_2\}^T[M]\{u_2\}\right) & 0 & \cdots \\ 0 & 0 & \left(\{u_3\}^T[M]\{u_3\}\right) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Modal Mass, Modal Damping, Modal Stiffness

The mass becomes

$$[U]^T[M][U]=\begin{bmatrix}\bar{m}_{11} & 0 & 0 & \cdots \\ 0 & \bar{m}_{22} & 0 & \cdots \\ 0 & 0 & \bar{m}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

The stiffness becomes

$$[U]^T[K][U]=\begin{bmatrix}\bar{k}_{11} & 0 & 0 & \cdots \\ 0 & \bar{k}_{22} & 0 & \cdots \\ 0 & 0 & \bar{k}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

The damping becomes (under special conditions)

$$[U]^T[C][U]=\begin{bmatrix}\bar{c}_{11} & 0 & 0 & \cdots \\ 0 & \bar{c}_{22} & 0 & \cdots \\ 0 & 0 & \bar{c}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

Modal Space Transformation

The physical set of highly coupled equations are transformed to modal space through the modal transformation equation to yield a set of uncoupled equations

Modal equations (uncoupled)

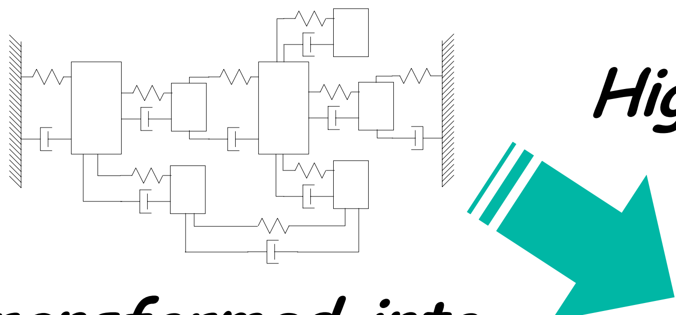
$$
\begin{bmatrix} \bar{m}_1 & & \\ & \bar{m}_2 & \\ & & \ddots \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \\ \vdots \end{Bmatrix} + \begin{bmatrix} \bar{c}_1 & & \\ & \bar{c}_2 & \\ & & \ddots \end{bmatrix} \begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \vdots \end{Bmatrix} + \begin{bmatrix} \bar{k}_1 & & \\ & \bar{k}_2 & \\ & & \ddots \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \{u_1\}^T \{F\} \\ \{u_2\}^T \{F\} \\ \vdots \end{Bmatrix}
$$

Modal Space Transformation

Diagonal Matrices -

Modal Mass | **Modal Damping** | **Modal Stiffness**

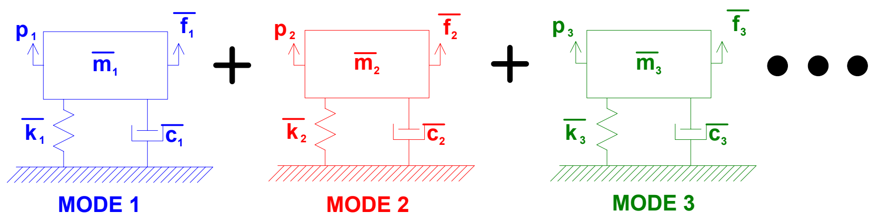
$$\begin{bmatrix} \ddots & & \\ & \overline{M} & \\ & & \ddots \end{bmatrix}\{\ddot{p}\} + \begin{bmatrix} \ddots & & \\ & \overline{C} & \\ & & \ddots \end{bmatrix}\{\dot{p}\} + \begin{bmatrix} \ddots & & \\ & \overline{K} & \\ & & \ddots \end{bmatrix}\{p\} = [U]^T\{F\}$$



Highly coupled system

transformed into

simple system



Modal Space Transformation

It is very clear to see that these modal space equations result in a set of independent SDOF systems

The modal transformation equation uncouples the highly coupled set of equations

The modal transformation appropriates the force to each modal oscillator in modal space

The modal transformation equation combines the response of all the independent SDOF systems to identify the total physical response

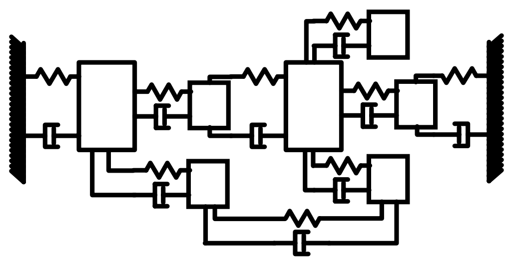
Modal Space Response Analysis

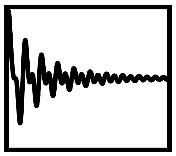
Since the MDOF system is reduced to equivalent SDOF systems with appropriate force, the response of each SDOF system can be determined using SDOF approaches discussed thus far.

The total response due to each of the SDOF systems can be determined using the modal transformation equation

Modal Space Transformation

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$





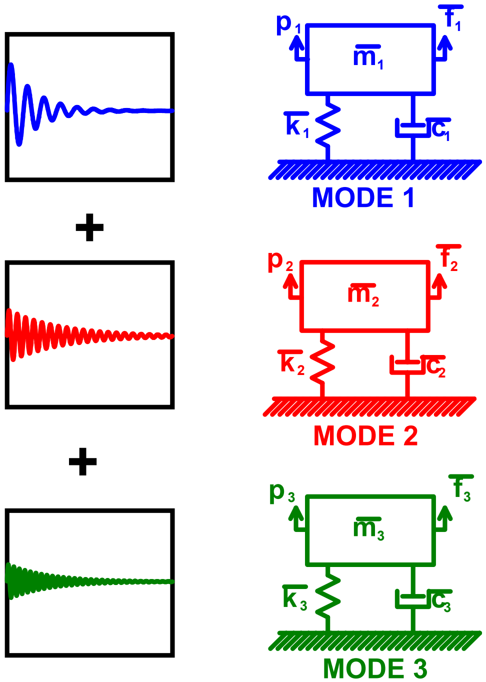
$= \Sigma$

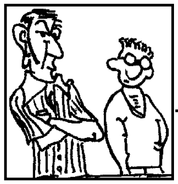
$$\{x\} = [U]\{p\} = [\{u_1\}\{u_2\}\{u_3\}\bullet\bullet\bullet]\{p\}$$

MODAL ⬇ **SPACE**

$$[\,\bar{M}\,]\{\ddot{p}\} + [\,\bar{C}\,]\{\dot{p}\} + [\,\bar{K}\,]\{p\} = [U]^T\{\bar{F}(t)\}$$

$$\{u_1\}p_1 + \{u_2\}p_2 + \{u_3\}p_3$$





Modal Space - Modal Matrices

The mode shapes are linearly independent and orthogonal w.r.t the mass and stiffness matrices

Modal Mass
$$[U_1]^T[M_1][U_1]=\begin{bmatrix}\ddots & & \\ & \overline{M}_1 & \\ & & \ddots\end{bmatrix}$$
TRUE !!!

Modal Damping
$$[U_1]^T[C_1][U_1]=\begin{bmatrix}\ddots & & \\ & \overline{C}_1 & \\ & & \ddots\end{bmatrix}$$
???????

Modal Stiffness
$$[U_1]^T[K_1][U_1]=\begin{bmatrix}\ddots & & \\ & \overline{K}_1 & \\ & & \ddots\end{bmatrix}$$
TRUE !!!

Proportional Damping

The damping matrix is only uncoupled for a special case where the damping is assumed to be proportional to the mass and/or stiffness matrices

$$[U_1]^T[\alpha[M]+\beta[K]][U_1]=\begin{bmatrix} \setminus & & \\ & \alpha\overline{M}+\beta\overline{K} & \\ & & \setminus \end{bmatrix}$$

Many times proportional damping is assumed since we do not know what the actual damping is

This assumption began back when computational power was limited and matrix size was of critical concern. But even today we still struggle with the damping matrix !!!

Non-Proportional Damping

If the damping is not proportional then a state space solution is required (beyond our scope here)

$$\begin{bmatrix} [0] & [M_1] \\ [M_1] & [C_1] \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \dot{x} \end{Bmatrix} - \begin{bmatrix} [M_1] & [0] \\ [0] & [-K_1] \end{bmatrix} \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

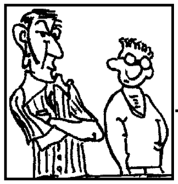
The equation of motion can be recast as

$$[B_1]\{\dot{Y}\} - [A_1]\{Y\} = \{Q\}$$

$$[B_1] = \begin{bmatrix} [0] & [M_1] \\ [M_1] & [C_1] \end{bmatrix} \qquad [A_1] = \begin{bmatrix} [M_1] & [0] \\ [0] & [-K_1] \end{bmatrix}$$

The eigensolution and modal transformation is then

$$[[A_1] - \lambda[B_1]]\{Y\} = \{0\} \qquad \{Y\} = [\Phi_1]\{p_1\}$$



MATLAB Examples - VTB4_1

```
VIBRATION TOOLBOX EXAMPLE 4_1

>> clear
>> m=[1 0;0 1];k=[2 -1;-1 1];
>> [P,w,U]=VTB4_1(m,k)

P =
        0.5257    0.8507
        0.8507   -0.5257
w =
        0.6180
        1.6180
U =
        0.5257    0.8507
        0.8507   -0.5257
>> m
m =
        1     0
        0     1
>> k
k =
        2    -1
       -1     1

>>
```

MATLAB Examples - VTB4_3

VIBRATION TOOLBOX EXAMPLE 4_3

```
>> clear
>> m=[1 0;0 1];d=[.1 0;0 .1];k=[2 -1;-1 1];
>> [v,w,zeta]=VTB4_3(m,d,k)
Damping is proportional, eigenvectors are real.

v =
     -0.5257   -0.8507
     -0.8507    0.5257
w =
     0.6180
     1.6180
zeta =
     0.0809
     0.0309
>> m
m =
     1     0
     0     1
>> d
d =
     0.1000         0
          0    0.1000
>> k
k =
     2    -1
    -1     1
```