

Vibration Mechanics Hw #2

(Free Vibration of SDOF Systems)

Issued: Tue, Mar. 07 2023

Due: Thu, Mar. 23 2023 18:00

1. Rao P. 2.7 natural frequency calculation

For small angular rotation of bar PQ about P,

$$\frac{1}{2} (k_{12})_{eq} (\theta l_3)^2 = \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_2)^2$$

$$\text{i.e., } (k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2) / l_3^2$$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

$$k_{eq} = \frac{(k_{12})_{eq} k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{ k_1 \left(\frac{l_1}{l_3} \right)^2 + k_2 \left(\frac{l_2}{l_3} \right)^2 \right\} k_3}{k_1 \left(\frac{l_1}{l_3} \right)^2 + k_2 \left(\frac{l_2}{l_3} \right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m (k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

2. Rao P. 2.16 natural frequency calculation

2.16 $x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$

Conservation of momentum:

$$(M+m)\dot{x}_0 = mv \text{ or } \dot{x}_0 = \dot{x}(t=0) = \frac{mv}{M+m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{4k}{M+m}}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2 g^2}{16k^2} + \frac{m^2 v^2}{(M+m)4k} \right\}^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right) = \tan^{-1} \left(-\frac{mg}{4k} \sqrt{\frac{4k}{(M+m)}} \frac{(M+m)}{mv} \right) = \tan^{-1} \left(-\frac{g\sqrt{M+m}}{v\sqrt{4k}} \right)$$

(a) Velocity of hammer = 15 m/s

Mass of hammer, $m = 6$ kg

Mass of anvil, $m = 50$ kg

$$A_0 = \left\{ \left(\frac{6 \times 9.81}{(4)(17.5 \times 10^3)} \right)^2 + \frac{(6^2)(15^2)}{(56)(70 \times 10^3)} \right\}^{\frac{1}{2}} = 0.0082 \text{ m or } 8.2 \text{ mm}$$

$$\phi_0 = \tan^{-1} \left\{ -\frac{9.81\sqrt{56}}{(15)\sqrt{70 \times 10^3}} \right\} = -1.06 \text{ degrees}$$

(b) $x = 0$ at static equilibrium position: $x_0 = x(t=0) = 0$. Conservation of momentum gives:

$$M\dot{x}_0 = mv \text{ or } \dot{x}_0 = \dot{x}(t=0) = \frac{mv}{M}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2 v^2 (M)}{M^2 4k} \right\}^{\frac{1}{2}} = \frac{mv}{\sqrt{4kM}} = \frac{(6)(15)}{\sqrt{(70 \times 10^3)(50)}} = 0.048 \text{ m or } 48 \text{ mm}$$

$$\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right) = \tan^{-1}(0) = 0$$

3. Rao P. 2.23 natural frequency calculation

- (a) Neglect masses of rigid links. Let x = displacement of W. Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2m}}$$

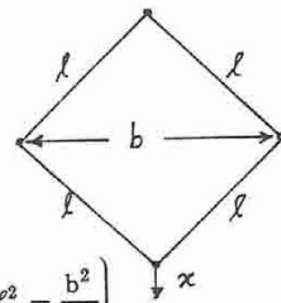
- (b) Under a displacement of x of mass, each spring will be compressed by an amount:

$$x_s = x \frac{2}{b} \sqrt{\ell^2 - \frac{b^2}{4}}$$

Equivalent spring constant:

$$\frac{1}{2} k_{eq} x^2 = 2 \left(\frac{1}{2} k x_s^2 \right)$$

$$\text{or } k_{eq} = 2 k \left(\frac{x_s}{x} \right)^2 = 2 k \left(\frac{4}{b^2} \right) \left(\ell^2 - \frac{b^2}{4} \right) = \frac{8 k}{b^2} \left(\ell^2 - \frac{b^2}{4} \right)$$



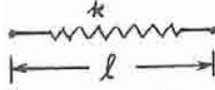
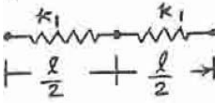
Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8 k}{b^2 m} \left(\ell^2 - \frac{b^2}{4} \right)}$$

4. Rao P. 2.40 natural frequency calculation

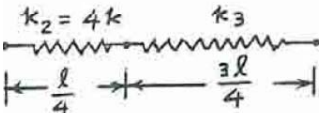



$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{total} = \frac{k_1}{2} \equiv k; k_1 = 2k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}}$$

$$0.5 = 2\pi \sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$


$$\frac{1}{k_{total}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

$$k_3 = \frac{4}{3}k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}} \text{ where } k_{eq} = 4k + \frac{4}{3}k = \frac{16}{3}k$$

$$\therefore \tau_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4} \sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4} \left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

5. Rao P. 2.46 Equation of motion

Consider the springs connected to the pulleys (by rope) to be in series. Then

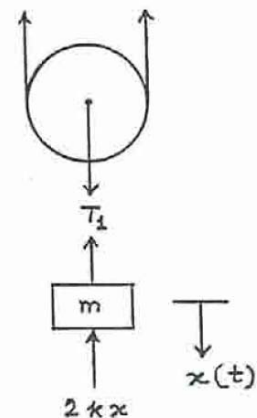
$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \text{ or } k_{eq} = \frac{5}{6}k$$

Let the displacement of mass m be x .

Then the extension of the rope (springs connected to the pulleys) = $2x$. From the free body diagram, the equation of motion of mass m :

$$m\ddot{x} + 2kx + k_{eq}(2x) = 0$$

$$\text{or } m\ddot{x} + \frac{11}{3}kx = 0$$



6. Rao P. 2.87 Torsional vibration

2.67 (a) $\omega_n = \sqrt{\frac{g}{l}}$

(b) $m l^2 \ddot{\theta} + \kappa a^2 \sin \theta + m g l \sin \theta = 0$; $m l^2 \ddot{\theta} + (\kappa a^2 + m g l) \theta = 0$
 $\omega_n = \sqrt{\frac{\kappa a^2 + m g l}{m l^2}}$

(c) $m l^2 \ddot{\theta} + \kappa a^2 \sin \theta - m g l \sin \theta = 0$; $m l^2 \ddot{\theta} + (\kappa a^2 - m g l) \theta = 0$
 $\omega_n = \sqrt{\frac{\kappa a^2 - m g l}{m l^2}}$

configuration (b) has the highest natural frequency.

7. Rao P. 2.90 Torsional vibration

$J_0 =$ mass moment of inertia of the ring $= 1.0 \text{ kg-m}^2$.
 $I_{os} =$ polar moment of inertia of the cross section of steel shaft
 $= \frac{\pi}{32} (d_{os}^4 - d_{is}^4) = \frac{\pi}{4} (0.05^4 - 0.04^4) = 36.2266 (10^{-8}) \text{ m}^4$

$I_{ob} =$ polar moment of inertia of cross section of brass shaft
 $= \frac{\pi}{32} (d_{ob}^4 - d_{ib}^4) = \frac{\pi}{32} (0.04^4 - 0.03^4) = 17.1806 (10^{-8}) \text{ m}^4$

$k_{ts} =$ torsional stiffness of steel shaft
 $= \frac{G_s I_{os}}{\ell} = \frac{(80 (10^9)) (36.2266 (10^{-8}))}{2} = 14490.64 \text{ N-m/rad}$

$k_{tb} =$ torsional stiffness of brass shaft
 $= \frac{G_b I_{ob}}{\ell} = \frac{(40 (10^9)) (17.1806 (10^{-8}))}{2} = 3436.12 \text{ N-m/rad}$

$k_{teq} = k_{ts} + k_{tb} = 17,926.76 \text{ N-m/rad}$

Torsional natural frequency:
 $\omega_n = \sqrt{\frac{k_{teq}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908 \text{ rad/sec}$

Natural time period:
 $\tau_n = \frac{2 \pi}{\omega_n} = \frac{2 \pi}{133.8908} = 0.04693 \text{ sec}$

Equation of motion

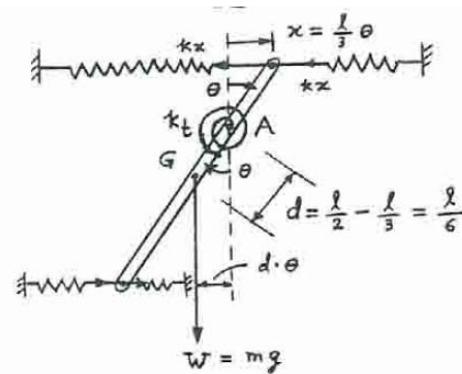
$$J_A \ddot{\theta} = -W d \theta - 2k \left(\frac{l}{3} \theta \right) \frac{l}{3} - 2k \left(\frac{2l}{3} \theta \right) \frac{2l}{3} - k_t \theta$$

where

$$J_A = J_G + m d^2 = \frac{1}{12} m l^2 + m \frac{l^2}{36} = \frac{1}{9} m l^2$$

$$\therefore \frac{m l^2}{9} \ddot{\theta} + (mg d + 2k \frac{l^2}{9} + \frac{8k l^2}{9} + k_t) \theta = 0$$

$$\omega_n = \sqrt{\frac{(mg d + \frac{2}{9} k l^2 + \frac{8}{9} k l^2 + k_t) 9}{m l^2}} = \sqrt{\frac{9mg d + 10k l^2 + 9k_t}{m l^2}}$$

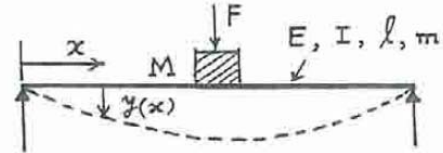


For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10(2000)(5)^2 + 9(1000)}{10(5)^2}} = 45.1547 \frac{\text{rad}}{\text{sec}}$$

9. Rao P. 2.106 Rayleigh method

Let m_{eff} = effective part of mass of beam (m) at middle. Thus vibratory inertia force at middle is due to $(M + m_{\text{eff}})$. Assume a deflection shape: $y(x, t) = Y(x) \cos(\omega_n t - \phi)$ where $Y(x)$ = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left(3 \frac{x}{\ell} - 4 \frac{x^3}{\ell^3} \right) ; 0 \leq x \leq \frac{\ell}{2}$$

where Y_0 = maximum deflection of the beam at middle = $\frac{F \ell^3}{48 E I}$

Maximum strain energy of beam = maximum work done by force $F = \frac{1}{2} F Y_0$.

Maximum kinetic energy due to distributed mass of beam:

$$\begin{aligned} &= 2 \left\{ \frac{1}{2} \frac{m}{\ell} \int_0^{\frac{\ell}{2}} \dot{y}^2(x, t) \Big|_{\max} dx \right\} + \frac{1}{2} (\dot{y}_{\max})^2 M \\ &= \frac{m \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y^2(x) dx + \frac{1}{2} \omega_n^2 Y_{\max}^2 M \\ &= \frac{m \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y_0^2 \left(\frac{9 x^2}{\ell^2} + 16 \frac{x^6}{\ell^6} - 24 \frac{x^4}{\ell^4} \right) dx + \frac{1}{2} Y_0^2 M \omega_n^2 \\ &= \frac{m \omega_n^2 Y_0^2}{\ell} \left[\frac{9}{\ell^2} \frac{x^3}{3} + \frac{16}{\ell^6} \frac{x^7}{7} - \frac{24}{\ell^4} \frac{x^5}{5} \right] \Big|_0^{\frac{\ell}{2}} + \frac{1}{2} Y_0^2 M \omega_n^2 \\ &= \frac{1}{2} Y_0^2 \omega_n^2 \left(\frac{17}{35} m + M \right) \end{aligned}$$

This shows that $m_{\text{eff}} = \frac{17}{35} m = 0.4857 m$

10. Rao P. 2.130 Viscous free vibration

(i) (a) Viscous damping, (b) Coulomb damping.

- (iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.
 (b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\frac{x_i}{x_{i+1}} = e^{\zeta \omega_n \tau_d}$

$$\ln \left(\frac{x_i}{x_{i+1}} \right) = \ln 2 = 0.6931 = \frac{2 \pi \zeta}{\sqrt{1 - \zeta^2}}$$

or $39.9590 \zeta^2 = 0.4804$ or $\zeta = 0.1096$

Since $\omega_d = \omega_n \sqrt{1 - \zeta^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m \omega_n^2 = \left(\frac{500}{9.81} \right) (31.6065)^2 = 5.0916 (10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2 m \omega_n}$$

Hence $c = 2 m \omega_n \zeta = 2 \left(\frac{500}{9.81} \right) (31.6065) (0.1096) = 353.1164 \text{ N-s/m}$

(b) From Eq. (2.135):

$$k = m \omega_n^2 = \frac{500}{9.81} (31.416)^2 = 5.0304 (10^4) \text{ N/m}$$

Using $N = W = 500 \text{ N}$,

$$\mu = \frac{0.002 k}{4 W} = \frac{(0.002) (5.0304 (10^4))}{4 (500)} = 0.0503$$

11. Rao P. 2.140 Equation of motion

Let δx = virtual displacement given to cylinder. Virtual work done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta}) (\delta\theta) - (m \ddot{x}) \delta x = -(J_0 \ddot{\theta}) \left(\frac{\delta x}{R}\right) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R} \left[\frac{\ddot{x}}{R} \right] - m \ddot{x} - k x - c \dot{x} = 0 \quad \text{or} \quad \frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0$$

Or

Newton's second law of motion:

$$\sum F = m \ddot{x} = -k x - c \dot{x} + F_f \quad (1)$$

$$\sum M = J_0 \ddot{\theta} = -F_f R \quad (2)$$

where F_f = friction force.

Using $J_0 = \frac{m R^2}{2}$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2R} \left(m R^2 \right) \frac{\ddot{x}}{R} = -\frac{1}{2} m \ddot{x} \quad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0 \quad (4)$$

The undamped natural frequency is: $\omega_n = \sqrt{\frac{2k}{3m}} \quad (5)$

