

Vibration Mechanics Hw #3

(Harmonically Excited Vibration)

10 Points each. Total 90 Pts.

Issued: Tue. Mar. 14, 2023

Due: Fri. April 07, 2023

1. Rao P. 3.16 Forced response of an undamped system

Equivalent stiffness of wing (beam) at location of engine:

$$k = \frac{\text{force}}{\text{deflection}} = \frac{3 E I}{\ell^3} = \frac{3 E \left(\frac{1}{12} b a^3 \right)}{\ell^3} = \frac{E b a^3}{4 \ell^3}$$

$$\text{Magnitude of unbalanced force: } = m r \omega^2 = m r \left(\frac{2 \pi N}{60} \right)^2 = \frac{m r \pi^2 N^2}{900}$$

$$\text{Equivalent mass of wing at location of engine: } M = \frac{33}{140} m_w = \frac{33}{140} (a b \ell \rho)$$

$$\text{Equation of motion: } M \ddot{x} + k x = m r \omega^2 \sin \omega t$$

Maximum steady state displacement of wing at location of engine:

$$X = \left| \frac{m r \omega^2}{k - M \omega^2} \right| = \left| \frac{\left(\frac{m r \pi^2 N^2}{900} \right)}{\left\{ \frac{E b a^3}{4 \ell^3} - \frac{33}{140} a b \ell \rho \left(\frac{2 \pi N}{60} \right)^2 \right\}} \right|$$

$$= \left| \frac{m r \ell^3 N^2}{22.7973 E b a^3 - 0.2357 \rho a b \ell^4 N^2} \right|$$

2. Rao P. 3.25 Forced response of an undamped system

Equation of motion for rotation about O:

$$J_0 \ddot{\theta} = -k \frac{\theta \ell}{4} \frac{\ell}{4} - k \frac{\theta 3 \ell}{4} \frac{3 \ell}{4} + M_0 \cos \omega t$$

$$\text{i.e., } J_0 \ddot{\theta} + \left(\frac{5}{8} k \ell^2 \right) \theta = M_0 \cos \omega t$$

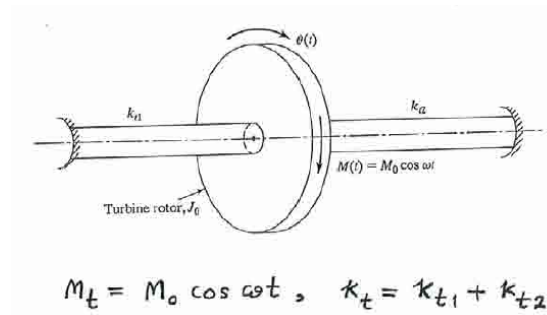
$$\text{where } J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4} \right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2$$

and $\omega = 1000 \text{ rpm} = 104.72 \text{ rad/sec}$. Steady state solution is:

$$\theta_p(t) = \Theta \cos \omega t$$

where

$$\Theta = \frac{M_0}{\frac{5}{8} k \ell^2 - J_0 \omega^2} = \frac{100}{5000 \left(\frac{5}{8} \right) (1^2) - 1.4583 (104.72^2)} = -0.007772 \text{ rad}$$



Equation of motion:

$$J_0 \ddot{\theta} + c_t \dot{\theta} + (k_{t1} + k_{t2}) \theta = M(t) = M_0 \cos \omega t \quad (1)$$

For the given data, Eq. (1) becomes

$$0.05 \ddot{\theta} + 2.5 \dot{\theta} + 7000 \theta = 200 \cos 500 t \quad (2)$$

Steady state response of the turbine rotor can be expressed, similar to Eqs. (3.25), (3.28) and (3.29) for a torsional system, as

$$\theta_p(t) = \Theta \cos(\omega t - \phi) \quad (3)$$

where

$$\Theta = \frac{M_0}{\left\{ (k_t - J_0 \omega^2)^2 + c_t^2 \omega^2 \right\}^{\frac{1}{2}}} \quad (4)$$

and

$$\phi = \tan^{-1} \left(\frac{c_t \omega}{k_t - J_0 \omega^2} \right) \quad (5)$$

For the given data,

$$J_0 = 0.05, M_0 = 200, k_t = 7000, c_t = 2.5, \omega = 500$$

Hence Eqs. (4) and (5) give

$$\begin{aligned} \Theta &= \frac{200}{\left[(7000 - 0.05 \times 25 \times 10^4)^2 + (2.5)^2 (25 \times 10^4) \right]^{\frac{1}{2}}} \\ &= 6.2868 \times 10^{-6} \text{ rad} \end{aligned}$$

$$\begin{aligned} \phi &= \tan^{-1} \left(\frac{2.5 \times 500}{7000 - 0.05 \times 250000} \right) \\ &= \tan^{-1} \left(-\frac{1250}{5500} \right) = \tan^{-1} (-0.2273) \\ &= -12.8043^\circ = -0.2235 \text{ rad} \end{aligned}$$

4. Rao P. 3.45

Forced response of a damped system

$$\frac{X}{Y} = \left[\frac{k^2 + c^2 \omega^2}{(k - m \omega^2)^2 + c^2 \omega^2} \right]^{\frac{1}{2}} \cdot \text{sp}$$

$$\text{or } \frac{10^{-6}}{Y} = \left[\frac{(10^6) + (10^3 (200 \pi))^2}{\left\{ 10^6 - \left(\frac{5000}{9.81} \right) (200 \pi)^2 \right\}^2 + \left\{ (10^3) (200 \pi) \right\}^2} \right]^{\frac{1}{2}}$$

$$\text{or } Y = 169.5294 (10^{-6}) \text{ m}$$

5. Rao P. 3.47

Forced response of a damped system

$m = 100 \text{ kg}$, $F_0 = 100 \text{ N}$, $X_{\max} = 0.005 \text{ m}$ at $\omega = 300 \text{ rpm} = 31.416 \text{ rad/sec}$.
Equations (3.33) and (3.34) yield:

$$\omega = \omega_n \sqrt{1 - 2 \zeta^2} = \sqrt{\frac{k}{m}} \sqrt{1 - 2 \zeta^2} = 31.416 \quad (1)$$

$$\text{or } k (1 - 2 \zeta^2) = (100) (31.416^2) = 98,696.5056$$

$$\text{and } X_{\max} = \delta_{st} \frac{1}{2 \zeta \sqrt{1 - \zeta^2}} = \frac{F_0}{k} \frac{1}{2 \zeta \sqrt{1 - \zeta^2}} = 0.005 \quad (2)$$

$$\text{or } k \zeta \sqrt{1 - \zeta^2} = \frac{F_0}{2 (0.005)} = 10,000.0$$

Divide Eq. (1) by (2):

$$\frac{1 - 2 \zeta^2}{\zeta \sqrt{1 - \zeta^2}} = 9.8696 \quad (3)$$

Squaring Eq. (3) and rearranging leads to:

$$101.4090 \zeta^4 - 101.4090 \zeta^2 + 1 = 0 \quad \text{or } \zeta = 0.0998, 0.9950$$

Using $\zeta = 0.0998$ in Eq. (1), we obtain

$$k = \frac{98696.5056}{1 - 2 (0.0998^2)} = 100,702.4994 \text{ N/m}$$

Since $\zeta = \frac{c}{2 m \omega_n}$, we find

$$c = 2 m \omega_n \zeta = 2 (100) \sqrt{\frac{100702.4944}{1000}} (0.0998) = 633.4038 \text{ N-s/m}$$

6. Rao P. 3.55

Response of a system under the base harmonic motion

$$\ddot{y}(t) = \ddot{x}_g(t) = A \cos \omega t ; \quad \dot{y}(t) = \frac{A}{\omega} \sin \omega t + B_1$$

$$y(t) = -\frac{A}{\omega^2} \cos \omega t + B_1 t + B_2$$

Assuming $y(0) = \dot{y}(0) = 0$, we get

$$y(t) = -\frac{A}{\omega^2} \cos \omega t$$

Equation of motion:

$$m\ddot{x} + k(x-y) = 0$$

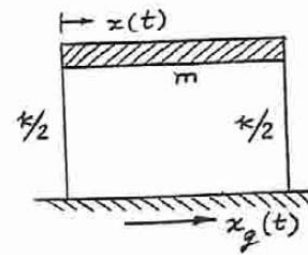
$$\text{i.e., } m\ddot{z} + kz = -m\ddot{y} = -m\ddot{x}_g(t) = -mA \cos \omega t$$

where $z = x - y$

Solution is:

$$z(t) = \frac{-mA \cos \omega t}{k - m\omega^2}$$

$$\therefore x(t) = z(t) + y(t) = -\left(\frac{m}{k - m\omega^2} + \frac{1}{\omega^2}\right) A \cos \omega t$$



7. Rao P. 3.75

Rotating unbalance

Let width = 0.5 m and thickness = t m.

$$I = \frac{1}{12} (0.5) t^3 = \frac{t^3}{24} \text{ m}^4$$

$$k = \frac{3EI}{l^3} = \frac{3(2.07 \times 10^{11}) \left(\frac{t^3}{24}\right)}{(5)^3} = 2.07 \times 10^8 t^3 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{M + 0.2357 m}}$$

$$\text{Where } m = \text{mass of beam} = (5 \times 0.5 \times t) \left(\frac{76.5 \times 10^3}{9.81}\right) = 19495.41 t \text{ kg}$$

$$\omega_n = \sqrt{\frac{2.07 \times 10^8 t^3}{75 + 0.2357 (19495.41 t)}}$$

$$r = \frac{\omega}{\omega_n} = 125.664 \sqrt{\frac{75 + 4595.0688 t}{2.07 \times 10^8 t^3}}$$

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k |r^2 - 1|}$$

$$\text{i.e., } 0.5 = \frac{5000}{(2.07 \times 10^8 t^3) \left\{ (125.664)^2 \left[\frac{75 + 4595.0688 t}{2.07 \times 10^8 t^3} \right] - 1 \right\}}$$

$$\text{i.e., } 1.3108 \times 10^4 t^3 - 4595.069 t - 74.367 = 0$$

By trial and error, the value of t is found as

$$t \approx 0.6 \text{ m.}$$

Since this is too large, we can start with a new width such as 1.0 m.

8. Rao P. 3.92 Coulomb damping

$$m = 25 \text{ kg}, \quad k = 10000 \text{ N/m}, \quad \mu = 0.3$$

$$\omega = 8 \text{ Hz} = 50.2656 \text{ rad/s}$$

$$X = 0.2 \text{ m}$$

Eg. (3.88) gives

$$c_{eq} = \frac{4 \mu N}{\pi \omega n} = \frac{4 (0.3) (25 \times 9.81)}{\pi (50.2656) (0.2)}$$
$$= 9.3183 \text{ N-s/m}$$

Note: It is assumed that friction force is small compared to F_0 in finding the equivalent viscous damping constant of the system.

Damping force = $F = c(\dot{x})^n$

Energy dissipated per quarter cycle during harmonic motion $x(t) = X \sin \omega t$ is

$$\frac{\Delta W}{4} = \int_0^{\pi/2\omega} c(\dot{x})^n dx = \int_0^{\pi/2\omega} c(\omega X \cos \omega t)^n dx$$

But $dx = \dot{x} dt = \omega X \cos \omega t \cdot dt$

$$\begin{aligned} \Delta W &= 4c \omega^{n+1} X^{n+1} \int_0^{\pi/2\omega} \cos^{n+1} \omega t \cdot dt \\ &= 4c \omega^{n+1} X^{n+1} \left\{ \frac{1}{(n+1)\omega} \cos^n \omega t \cdot \sin \omega t \Big|_0^{\pi/2\omega} + \frac{n}{n+1} \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \right\} \\ &= 4c \omega^{n+1} X^{n+1} \left(\frac{n}{n+1} \right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \end{aligned}$$

Equating this expression to $\pi c_{eq} \omega X^2$, we obtain

$$c_{eq} = \frac{4c \omega^n X^{n-1}}{\pi} \left(\frac{n}{n+1} \right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \equiv c \omega^n X^{n-1} \alpha_n$$

where $\alpha_n = \frac{4}{\pi} \left(\frac{n}{n+1} \right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \quad \text{--- (E}_1\text{)}$

For example, for $n=2$, (E₁) becomes

$$\alpha_n = \frac{4}{\pi} \left(\frac{2}{3} \right) \int_0^{\pi/2\omega} \cos \omega t \cdot dt = \frac{8}{3\pi} \left(\frac{\sin \omega t}{\omega} \right) \Big|_0^{\pi/2\omega} = \frac{8}{3\pi\omega}$$

and hence $c_{eq} = \frac{8c\omega X}{3\pi}$

which can be seen to be same as the expression found in Example 3.7.

For few other values of n , α_n can be found as follows:

n	1	2	3	4
α_n	$\frac{1}{\omega}$	$\frac{8}{3\pi\omega}$	$\frac{3}{4\omega}$	$\frac{32}{15\pi\omega}$

The amplitude can be found as

$$\begin{aligned} X &= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1 - r^2)^2 + c_{eq}^2 \omega^2}} \\ &= \frac{F_0}{\sqrt{k^2(1 - r^2)^2 + c^2 \omega^{2(n+1)} X^{2(n-1)} \alpha_n^2}} \end{aligned}$$