Vibration Mechanics Hw #4

(General Condition Vibration)

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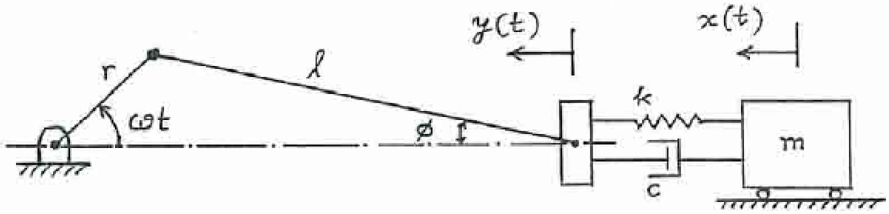
1. Rao P. 4.7 General periodic loading



Base motion is given by:

$$y(t) = r + \ell - r\cos\omega t - \ell\cos\phi = r + \ell - r\cos\omega t - \ell\sqrt{1 - \sin^2\phi} \quad (1)$$

Using $\ell\sin\phi = r\sin\omega t$, Eq. (1) becomes

$$y(t) = r + \ell - r\cos\omega t - \ell\sqrt{1 - \frac{r^2}{\ell^2}\sin^2\omega t} \quad (2)$$

Using the approximation:

$$\sqrt{1 - \frac{r^2}{\ell^2}\sin^2\omega t} \approx 1 - \frac{r^2}{2\,\ell^2}\sin^2\omega t \quad (3)$$

Eq. (2) can be expressed as

$$\begin{aligned} y(t) &= r + \ell - r\cos\omega t - \ell\left(1 - \frac{1}{2}\frac{r^2}{\ell^2}\sin^2\omega t\right) \\ &= r - r\cos\omega t + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2\cos 2\omega t \end{aligned} \quad (4)$$

Equation of motion:

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx &= ky + c\dot{y} \\ &= kr - kr\cos\omega t + \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2\cos 2\omega t + \cdots \\ &\quad + c\,r\,\omega\sin\omega t + \frac{c\,\ell}{4}\left(\frac{r}{\ell}\right)^2(2\,\omega)\sin 2\,\omega t + \cdots \end{aligned} \quad (5)$$

Solution of Eq. (5) can be found by adding the solutions due to each term on the right hand side of Eq. (5).

Solution due to constant term, F_0 (terms 1 and 3 on the r.h.s. of Eq. (5)):

$$x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\varsigma^2}}\, e^{-\varsigma \omega_n t} \cos\left(\omega_d\, t - \phi\right)\right] \tag{6}$$

$$\text{where} \quad \phi = \tan^{-1}\left(\frac{\varsigma}{\sqrt{1-\varsigma^2}}\right)$$

Solution due to sinusoidal term, $F_0 \sin \Omega t$ (terms 5 and 6 on the r.h.s. of Eq. (5)):

$$x(t) = X \sin\left(\Omega\, t - \phi_0\right) \tag{7}$$

$$\text{where} \quad X = \frac{F_0}{\left[(k - m\,\Omega^2)^2 + c^2\,\Omega^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{c\,\Omega}{k - m\,\Omega^2}\right) \tag{8}$$

Solution due to cosine term, $F_0 \cos \Omega t$ (terms 2 and 4 in Eq. (5)):

$$x(t) = X \cos\left(\Omega\, t - \phi_0\right) \tag{9}$$

$$\text{where} \quad X = \frac{F_0}{\left[(k - m\,\Omega^2)^2 + c^2\,\Omega^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{c\,\Omega}{k - m\,\Omega^2}\right) \tag{10}$$

For given data, $\varsigma = \dfrac{c}{2\sqrt{m\,k}} = \dfrac{10}{2\sqrt{1\,(100)}} = 0.5$, $\dfrac{r}{\ell} = 0.1$, $\omega = 100$, $2\,\omega = 200$, etc. and the solution of Eq. (5) can be obtained by using Eqs. (6) to (8) suitably.

2. Rao P. 4.21 Duhamel integral (solve the case of Fig. 4.46(b) only)

$$F(\tau) = \begin{cases} F_0\left(\frac{\tau}{t_0}\right) & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

For $0 \le t \le t_0$: $\quad x(t) = \frac{F_0}{m\omega_n t_0}\int_0^t \tau \sin\omega_n(t-\tau)\, d\tau$

i.e. $$x(t) = \frac{F_0}{m\omega_n t_0}\left[\int_0^t (t-\tau)\sin\omega_n(t-\tau)(-d\tau) - t\int_0^t \sin\omega_n(t-\tau)(-d\tau)\right]$$

$$= \frac{F_0}{m\omega_n t_0}\left[\frac{1}{\omega_n^2}\sin\omega_n(t-\tau) - \frac{(t-\tau)}{\omega_n}\cos\omega_n(t-\tau)\right]_{\tau=0}^{t}$$

$$+ \frac{F_0\, t}{m\omega_n t_0}\left[\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k t_0}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n t_0}\int_0^{t_0} \tau \sin\omega_n(t-\tau)\, d\tau$$

$$= \frac{F_0}{m\omega_n t_0}\left[\frac{1}{\omega_n^2}\sin\omega_n(t-\tau) - \frac{(t-\tau)}{\omega_n}\cos\omega_n(t-\tau)\right]_{\tau=0}^{t}$$

$$+ \frac{F_0\, t}{m\omega_n t_0}\left[\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k t_0}\left[\frac{1}{\omega_n}\sin\omega_n(t-t_0) + t_0\cos\omega_n(t-t_0) - \frac{1}{\omega_n}\sin\omega_n t\right]$$

3. Rao P. 4.32 Forced response of a damped system

Equation of motion for rotation about O:

$$J_0\,\ddot\theta + M\,\ddot x\,(\ell) + k_1\,a^2\,\theta + k_2\,b^2\,\theta = F_0\,\ell\,e^{-t} \tag{1}$$

where $\ddot x = \ell\,\ddot\theta$ and

$$J_0 = \frac{1}{12}\,m\,\ell^2 + m\,(\frac{\ell}{2})^2 = \frac{1}{3}\,m\,\ell^2$$

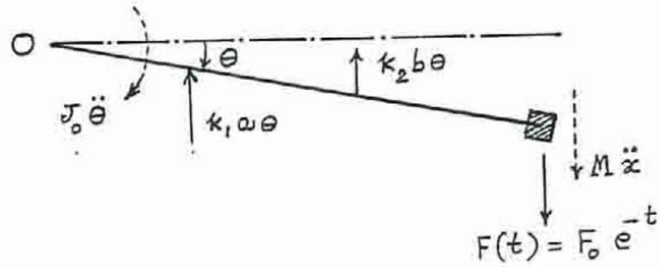
Eq. (1) can be rewritten as:

$$\left(\frac{1}{3}\,m\,\ell^2 + M\,\ell^2\right)\ddot\theta + (k_1\,a^2 + k_2\,b^2)\,\theta = F_0\,\ell\,e^{-t} \tag{2}$$

For given data, Eq. (2) takes the form:

$$53.3333\,\ddot\theta + 1562.5\,\theta = 500\,e^{-t} \tag{3}$$

Noting that the system is undamped with $\omega_n = \sqrt{\dfrac{1562.5}{53.3333}} = 5.4127$ rad/sec and



the forcing term as $500\,e^{-t}$, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$\begin{aligned}\theta(t) &= \frac{1}{(53.3333)\,(5.4127)}\int_0^t 500\,e^{-\tau}\,\sin 5.4127\,(t-\tau)\,d\tau \\ &= 1.7320\int_0^t e^{-t}\,e^{(t-\tau)}\,\sin 5.4127\,(t-\tau)\,d\tau \\ &= -\,1.7320\,e^{-t}\int_0^t e^{(t-\tau)}\,\sin 5.4127\,(t-\tau)\,(-\,d\tau)\end{aligned} \tag{4}$$

Using the formula:

$$\int e^{a\,x}\,\sin b\,x\,dx = \frac{1}{a^2+b^2}\,e^{a\,x}\left[a\,\sin b\,x - b\,\cos b\,x\right] \tag{5}$$

Eq. (4) can be expressed as

$$\begin{aligned}\theta(t) &= -\,1.7320\,e^{-t}\left[\frac{e^{(t-\tau)}}{1^2+5.4127^2}\left\{\sin 5.4127\,(t-\tau) - 5.4127\,\cos 5.4127\,(t-\tau)\right\}\right]_{\tau=0}^{\tau=t} \\ &= 0.3094\,e^{-t} + 0.05717\,\sin 5.4127\,t - 0.3094\,\cos 5.4127\,t \text{ radian}\end{aligned}$$

4. Rao P. 4.38 Fighter example

The stiffness of the cantilever beam (wing) is given by

$$k = \frac{3EI}{l^3} = \frac{3\,(15 \times 10^9)}{10^3} = 45 \times 10^6 \text{ N/m}$$

System can be modeled as a single degree of freedom undamped system:

$$m\ddot{x} + kx = 0$$

where $m = 2500$ kg, $k = 45 \times 10^6$ N/m, and

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45 \times 10^6}{2.5 \times 10^3}} = 134.1641 \text{ rad/s}$$

Response of mass due to impulse $\underset{\sim}{F}$ is given by Eq. (4.26) with $\zeta = 0$ and $\omega_d = \omega_n$:

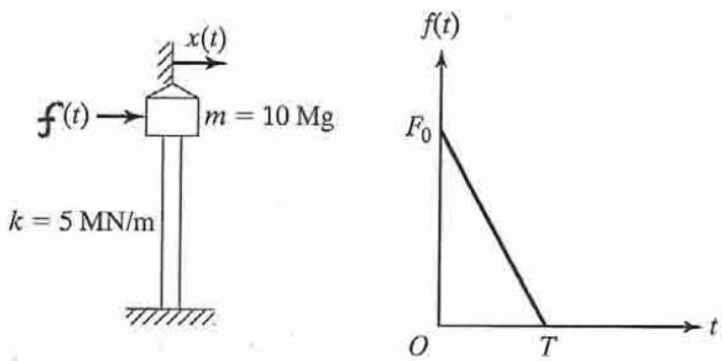
$$x(t) = \frac{\underset{\sim}{F}}{m\,\omega_n} \sin \omega_n t$$

$$= \frac{50}{2500\,(134.1641)} \sin 134.1641\, t$$

$$= 0.000149071 \sin 134.1641\, t \quad \text{m}$$

5. Rao P. 4.46 Response of a system by a general input

(4.46)



Equation of motion:

$$m\ddot{x} + kx = f(t) = \begin{cases} F_0\left(1 - \frac{t}{T}\right) & ; \quad 0 \le t \le T \\ 0 & ; \quad t > T \end{cases} \qquad (1)$$

Using Eqs. (E.8) and (E.9) in the solution of Example 4.13, the response of the water tank can be found as

$$x(t) = \begin{cases} \frac{F_0}{k}\left[1 - \frac{t}{T} - \cos\omega_n t + \frac{1}{\omega_n T}\sin\omega_n t\right] ; & 0 \le t \le T \\ \frac{F_0}{k\omega_n T}\left[(1-\cos\omega_n T)\sin\omega_n t - (\omega_n T - \sin\omega_n T)\cos\omega_n t\right] ; & t > T \end{cases} \qquad (2)$$

6. Rao P. 4.58 Response spectrum

$k = 5000$ N/m , $\zeta = 0.05$, $m = 250$ kg

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5000}{250}} = 4.4721 \text{ rad/s}$$

Natural time period = $\tau = \dfrac{2\pi}{\omega_n} = \dfrac{1}{f_n} = 1.4050$ s

(a) Maximum relative displacement of the transformer:

$$x_{max} = S_d \simeq 3.8 \text{ in} = 9.562 \text{ cm}$$

(b) Maximum shear force in the pole:

$$F_{max} = |k\, x_{max}| = m\, \ddot{x}_{max} = m\, S_a = m\,(0.2\, g)$$

$$= 250\ (0.2)\ (9.81) = 490.5 \text{ N}$$

(c) Maximum bending moment in the pole:

$$M_{max} = F_{max}\, l = 490.5\, l \text{ N-m}$$

Where l is the length of the pole (height of the transformer in meters.

7. Rao P. 4.65 Impact responses

From Example 4.23, the impulse applied to the mass M by the bronze ball is given by the right hand side of Eq. (E.9) of Example 4.23:

$$\underset{\sim}{F} = \frac{2\, m_0\, M}{m_0 + M}\; v_1\, \delta(t) \qquad (1)$$

with v_1 given by

$$m g h = \frac{1}{2}\, m v_1^2$$

i.e., $v_1 = \sqrt{2gh} = \sqrt{2\,(9.81)\,(2)} = 6.2642$ m/s,

$M = 2$ kg and $m_0 = 0.1$ kg.

Hence Eq. (1) becomes

$$\underset{\sim}{F} = \frac{2\,(0.1)\,(2)}{0.1+2}\,(6.2642)\;\delta(t) = 1.1932\;\delta(t)$$

The equivalent initial velocity due to the impulse $\underset{\sim}{F}$ is given by

$$\dot{x}_0 = \frac{\underset{\sim}{F}}{M} = \frac{1.1932}{2} = 0.5966 \text{ m/s}$$

For the damped spring-mass system,

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{Mk}} = \frac{5}{2\sqrt{2\,(100)}} = 0.1768$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{100}{2}} = 7.0711 \text{ rad/s}$$

$$\omega_d = \sqrt{1-\zeta^2}\;\omega_n = \sqrt{1-0.1768^2}\;(7.0711) = 6.9597 \text{ rad/s}$$

The free vibration response under the initial conditions $x(0) = 0$ and $\dot{x}_0 = 0.5966$ m/s can be found from Eq. (2.72):

$$x(t) = e^{-\zeta\omega_n t}\left\{ x_0 \cos \omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin \omega_d t \right\}$$

$$= e^{-0.1768\,(7.0711)\,t}\left\{ \frac{0.5966}{6.9597} \sin 6.9597\, t \right\}$$

$$\therefore x(t) = e^{-1.2502\, t}\,(0.08572)\sin 6.9597\, t \quad \text{m}$$