Vibration Mechanics Hw #7

(Continuous Systems)

Issued: May 26, 2023

Due: June 12, 2023

1. Rao P. 8.5

8.5 At $x = 0$: $$P \frac{\partial w}{\partial x} = m \frac{\partial^2 w}{\partial t^2} \qquad (E_1)$$

At $x = l$: $$P \frac{\partial w}{\partial x} = -k\, w \qquad (E_2)$$

General solution is

$$w(x,t) = W(x) \cdot T(t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right) \qquad (E_3)$$

Equations (E_1) and (E_3) give:

$$\Big\{P\left(-A \frac{\omega}{c} \sin \frac{\omega x}{c} + B \frac{\omega}{c} \cos \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$
$$= -m\omega^2 \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)\Big\}_{x=0}$$

i.e., $$A\,(m\omega^2) + B\left(\frac{P\omega}{c}\right) = 0 \qquad (E_4)$$

Eqs. (E_2) and (E_3) yield

$$P\left(-\frac{\omega}{c} A \sin \frac{\omega l}{c} + B \frac{\omega}{c} \cos \frac{\omega l}{c}\right) = -k\left(A \cos \frac{\omega l}{c} + B \sin \frac{\omega l}{c}\right)$$

i.e.,

$$A\left(-\frac{P\omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c}\right) + B\left(\frac{P\omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c}\right) = 0 \qquad (E_5)$$

Eqs. (E_4) and (E_5) give the frequency equation:

$$\begin{vmatrix} (m\omega^2) & (P\omega/c) \\ \left(-\frac{P\omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c}\right) & \left(\frac{P\omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c}\right) \end{vmatrix} = 0$$

which, upon simplification, becomes

$$\tan \alpha = \left\{ \frac{P k - \left(\frac{P m c^2}{l^2}\right)\alpha^2}{\left(\frac{c^2 m k}{l}\right)\alpha + \left(\frac{P^2 c}{l}\right)\alpha} \right\} \qquad (E_6)$$

2. Rao P. 8. 10

(8.10) $l = 2000$ m, $\rho = 8890$ kg/m^3, $0 \le \omega_1$ to $\omega_4 \le 20$ Hz

$$\omega_n = \frac{n c \pi}{l} = \frac{n\pi}{l}\sqrt{\frac{P}{\rho}}$$

For $\omega_4 \le 20\,(2\pi)$ rad/sec, $\frac{4\pi}{l}\sqrt{\frac{P}{\rho}} \le 40\pi$

i.e., $\sqrt{\frac{P}{8890}} \le \frac{40\pi\,(2000)}{4\pi}$

i.e., $P \le 35560 \times 10^8$ N

Let the permissible (yield) stress of the material be 300 MPa $= 3 \times 10^8$ N/m^2.

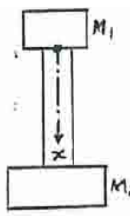
Assuming the diameter of cable as 0.1 m, area of cross-section is $\frac{\pi}{4}(0.1)^2 = 0.007854$ m^2, and the permissible tension (with factor of safety of one) is

$(0.007854)(3 \times 10^8) = 2,356,200$ N

$\therefore$ Initial tension $= 2.3562 \times 10^6$ N

3. Rao P. 8.17 (a)

(8.17) (a) $u(x,t) = \left(\underset{\sim}{A} \cos\frac{\omega x}{c} + \underset{\sim}{B} \sin\frac{\omega x}{c}\right)\left(C \cos\omega t + D \sin\omega t\right)$



At $x=0$: $M_1 \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial u}{\partial x}$

$$-\omega^2 M_1 \underset{\sim}{A} = AE \frac{\omega}{c} \underset{\sim}{B} \Rightarrow \underset{\sim}{B} = -\left(\frac{\omega M_1 c}{AE}\right)\underset{\sim}{A}$$

At $x = l$: $M_2 \frac{\partial^2 u}{\partial t^2} = -AE \frac{\partial u}{\partial x}$

$$-\omega^2 M_2\left(\underset{\sim}{A} \cos\frac{\omega l}{c} + \underset{\sim}{B} \sin\frac{\omega l}{c}\right) = -AE\frac{\omega}{c}\left(-\underset{\sim}{A}\sin\frac{\omega l}{c} + \underset{\sim}{B}\cos\frac{\omega l}{c}\right)$$

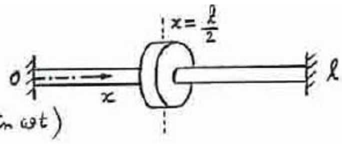
i.e. $\underset{\sim}{A}\left(-\omega^2 M_2 \cos\frac{\omega l}{c} - AE\frac{\omega}{c}\sin\frac{\omega l}{c}\right) = \underset{\sim}{B}\left(-\frac{AE\omega}{c}\cos\frac{\omega l}{c} + \omega^2 M_2 \sin\frac{\omega l}{c}\right)$

i.e. $\omega^2 M_2 \cos\frac{\omega l}{c} + \frac{AE\omega}{c}\sin\frac{\omega l}{c} + \frac{\omega M_1 c}{AE}\left(-\omega^2 M_2 \sin\frac{\omega l}{c} + \frac{AE\omega}{c}\cos\frac{\omega l}{c}\right) = 0$

This is the frequency equation.

4. Rao P. 8.22

(8.22) Consider half the shaft and half the disc.



$$\theta(x,t) = \left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)$$

$$\theta(0,t) = 0 \quad \Rightarrow \quad A = 0$$

$$GJ\,\frac{\partial\theta}{\partial x}\left(\frac{l}{2},t\right) = -\frac{J_0}{2}\,\frac{\partial^2\theta}{\partial t^2}\left(\frac{l}{2},t\right)$$

$$\Rightarrow \quad GJ\,\frac{\omega}{c}\cdot B\cos\frac{\omega l}{2c} = \frac{J_0\,\omega^2}{2}\cdot B\sin\frac{\omega l}{2c}$$

$$\Rightarrow \quad \tan\frac{\omega l}{2c} = \frac{2GJ}{J_0\,\omega c} = \frac{2c}{\omega l}\cdot\frac{GJl}{J_0 c^2}$$

Frequency equation: $\quad \alpha\tan\alpha = \beta \quad$ where $\alpha = \frac{\omega l}{2c}$ and $\beta = \frac{GJl}{J_0 c^2}$ $= \left(\frac{J\rho l}{J_0}\right)$

If roots of this equation are given by α_n,

$\omega_n = \frac{2\alpha_n c}{l}$ and

$$\theta(x,t) = \sum_{n=1}^{\infty}\sin\frac{\omega_n x}{c}\left(C_n\cos\omega_n t + D_n\sin\omega_n t\right) \qquad ---- (E_1)$$

$$\theta(x,0) = \sum \sin\frac{\omega_n x}{c}\left(C_n\right) = 0 \quad\Rightarrow\quad C_n = 0 \qquad ---- (E_2)$$

$$\dot\theta(x,0) = \sum_{n=1}^{\infty}\sin\frac{\omega_n x}{c}\left(\omega_n D_n\right) = \frac{2\,\dot\theta_0\,x}{l} \qquad ---- (E_3)$$

as $\dot\theta\Big|_{\frac{l}{2},\,t=0} = \dot\theta_0$ and hence $\dot\theta(x,0) = \frac{2\,\dot\theta_0\,x}{l}$

Multiply Eq. (E_3) by $\sin\frac{\omega_n x}{c}$ and integrate from 0 to $\frac{\pi c}{2\omega_n}$:

$$\omega_n D_n\int_0^{\left(\frac{\pi c}{2\omega_n}\right)}\sin^2\frac{\omega_n x}{c}\,dx = \frac{2\,\dot\theta_0}{l}\int_0^{\left(\frac{\pi c}{2\omega_n}\right)} x\,\sin\frac{\omega_n x}{c}\,dx \qquad (E_4)$$

i.e.,
$$D_n = \frac{8\,c\,\dot\theta_0}{\pi\,l\,\omega_n^2} \qquad (E_5)$$

$$\therefore \quad \theta(x,t) = \sum_{n=1}^{\infty}\left(\frac{8\,c\,\dot\theta_0}{\pi l\,\omega_n^2}\right)\sin\frac{\omega_n x}{c}\,\sin\omega_n t \qquad (E_6)$$

5. Rao P. 8.33

8.33

$$w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = w(l,t) = \frac{\partial^2 w}{\partial x^2}(l,t) = 0 \; ; \quad t \geq 0$$

$$W(x) = C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x)$$
$$+ C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$\frac{d^2 W}{dx^2}(x) = C_1\beta^2(-\cos\beta x + \cosh\beta x) + C_2\beta^2(-\cos\beta x - \cosh\beta x)$$
$$+ C_3\beta^2(-\sin\beta x + \sinh\beta x) + C_4\beta^2(-\sin\beta x - \sinh\beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$W(l) = 0 \Rightarrow C_3(\sin\beta l + \sinh\beta l) + C_4(\sin\beta l - \sinh\beta l) = 0 \quad \text{-- } (E_1)$$

$$\frac{d^2 W}{dx^2}(l) = 0 \Rightarrow C_3(-\sin\beta l + \sinh\beta l) - C_4(\sin\beta l + \sinh\beta l) = 0 \quad \text{-- } (E_2)$$

Setting the determinant of the coefficient matrix of C_3 and C_4 in (E_1) and (E_2) to zero, we get the frequency equation

$$-(\sin\beta l + \sinh\beta l)^2 + (\sin\beta l - \sinh\beta l)^2 = 0$$

or $\quad \sin\beta l \; \sinh\beta l = 0$

Since $\sinh\beta l \neq 0$, the frequency equation becomes

$$\sin\beta l = 0$$

$$\therefore \beta_n l = n\pi \; ; \quad \omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}} \; ; \; n = 1, 2, \cdots$$

6. Rao P. 8.36

8.36

From the solution of problem 8.31, we find for $\sin\beta l = 0$,

$$C_3 = C_4$$

Hence $\quad W_n(x) = C_n \sin\beta_n x = C_n \sin\frac{n\pi x}{l}$

General free vibration solution is

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}(A_n\cos\omega_n t + B_n\sin\omega_n t)$$

When beam vibrates in the first mode,

$$w(x,t) = \sin\frac{\pi x}{l}(A_1\cos\omega_1 t + B_1\sin\omega_1 t)$$

Bending moment in the beam is

$$M(x,t) = EI\frac{\partial^2 w}{\partial x^2}(x,t) = -\frac{\pi^2 EI}{l^2}\sin\frac{\pi x}{l}(A_1\cos\omega_1 t + B_1\sin\omega_1 t)$$

$$= -\frac{\pi^2}{l^2}EI\, w(x,t)$$

If the amplitude of vibration is Y, the maximum bending moment is $\quad |M_{max}| = \frac{\pi^2}{l^2}EI\,Y$

For the given data,

$$|M_{max}| = \frac{\pi^2}{(1)^2}(20.5\times10^{10})\left(\frac{10^3}{10^{12}}\right)\left(\frac{10}{1000}\right) = 20.2328 \text{ N-m}$$