chapter

4

Analytical Mechanics: Basic Concepts

4.1 Introduction

This chapter and Chapter 5 introduce analytical techniques for describing the motion of dynamical systems. The dynamical system is considered as a whole and scalar quantities such as energy and work are used. Constraint forces and moments are treated differently than in Newtonian mechanics. Constraint forces that do no work do not appear in the formulation, and they are accounted for by appropriately selecting the variables used to describe the motion. Sometimes, one may need to find out the magnitudes of the constraint forces. This can be accomplished by calculating the magnitudes of the constraint forces after the problem is solved, or by leaving the constraints in the system formulation by means of Lagrange multipliers. The approaches described in this chapter are analytical approaches and they are based on the principles of variational calculus. Appendix B provides a more detailed look at variational principles. Generalized coordinates, which do not necessarily have to be physical coordinates, are used as motion variables. This makes the analytical approach more flexible than the Newtonian, as the Newtonian approach is implemented using physical coordinates.

We derive the analytical equations of motion in this chapter for particles and for plane motion of rigid bodies, though these equations are valid for three-dimensional rigid body motion and deformable bodies as well. Chapter 8 will deal with D'Alembert's principle and Lagrange's equations for the general three-dimensional motion of rigid bodies.

One question often asked is whether it is more convenient to use a Newtonian technique or an analytical one when obtaining the equations of motion. There is no set answer to this question, with the possible exception of dynamical systems consisting of several interconnected components. When the number of coordinates needed to describe the system is much less than the number of components, it is usually preferable to use analytical techniques. When amplitudes of reaction forces are sought, it is usually better to use a Newtonian analysis. Looking at the problem from *both* the Newtonian and analytical points of view gives one more insight and a better understanding.

Analytical techniques use scalar functions like work and energy in the formulation, rather than vector quantities. While this approach makes a lot of sense, the experiences of dynamicists in recent years have shown that vector approaches combined with analytical techniques are more desirable when modeling complex systems. One advantage of a vector approach is that it can be implemented on a digital computer more readily.

4.2 GENERALIZED COORDINATES

A system of N particles requires $3N$ physical coordinates to specify the system's position. Consider an inertial coordinate system and let the vector $\mathbf{r}_i = \mathbf{r}_i(x_i, y_i, z_i)$ be the mapping of the ith particle in this coordinate system.[1] We express $\mathbf{r}_i$ as (Fig. 4.1)

$$\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k} \qquad i = 1, 2, \ldots, N \tag{4.2.1}$$

The $3N$ coordinates required to represent the system span a $3N$-dimensional space, which is called the *configuration space* of dimension $n = 3N$. In many cases, as we will soon see, it is more advantageous to use a different set of variables than the physical coordinates to describe the motion. This approach is analogous to that of using different coordinate systems that we saw in Chapter 1. We introduce a set of variables $q_1, q_2, \ldots, q_n$, related to the physical coordinates by

$$\begin{aligned} x_1 &= x_1(q_1, q_2, \ldots, q_n) \\ y_1 &= y_1(q_1, q_2, \ldots, q_n) \\ z_1 &= z_1(q_1, q_2, \ldots, q_n) \\ x_2 &= x_2(q_1, q_2, \ldots, q_n) \\ &\vdots \\ z_n &= z_n(q_1, q_2, \ldots, q_n) \end{aligned} \tag{4.2.2}$$

We will refer to a set of variables that can completely describe the position of a dynamical system as *generalized coordinates*. The space spanned by the generalized coordinates is the *configuration space*. As an illustration, consider the spherical pendulum in Fig. 4.2, whose length can change. The motion of the pendulum can be described by the Cartesian coordinates x, y, and z, or by q_1, q_2, and q_3, where $q_1 = L$ describes the length of the pendulum, and $q_2 = \theta$ and $q_3 = \phi$ describe the angular

[1] If a noninertial coordinate system is used, one has to include the variables describing the motion of the reference frame in the set of coordinates that describe the motion, unless the characteristics of the reference frame are treated as known.

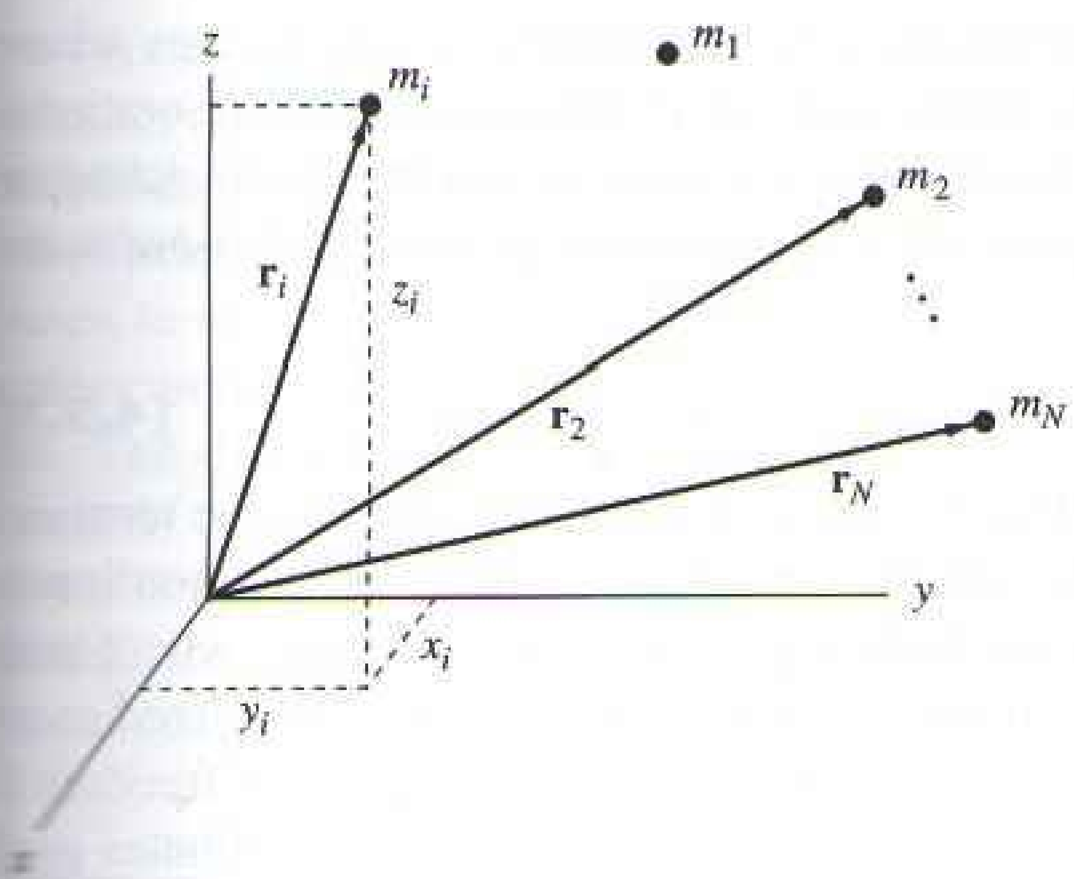

Figure 4.1 A system of N particles

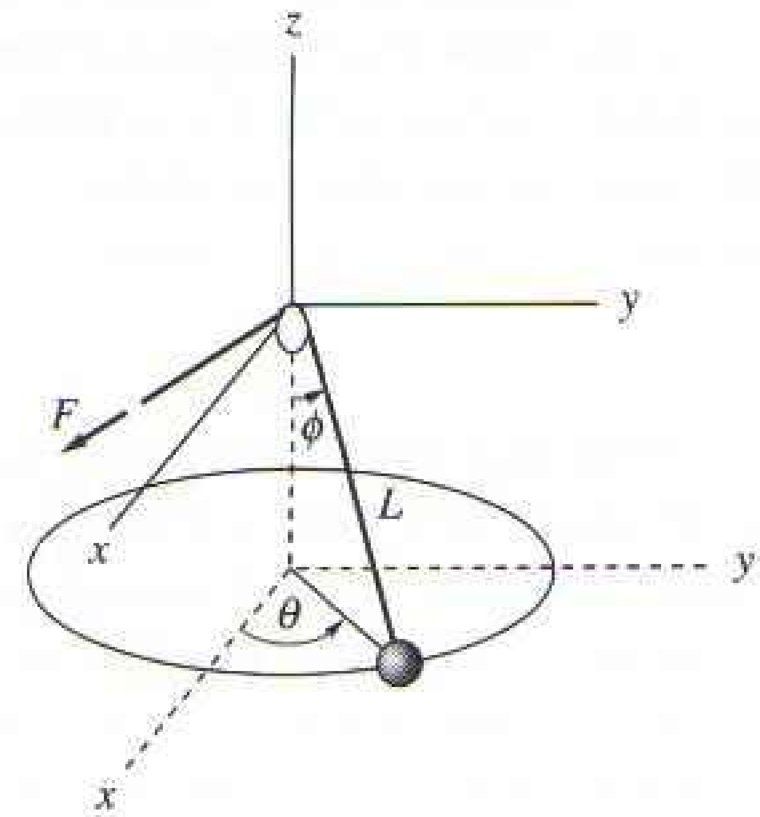

Figure 4.2 A spherical pendulum whose length changes

displacement. The choice of L, θ, and ϕ as generalized coordinates is equivalent to using spherical coordinates. The two sets of coordinates are related by

$$x = q_1 \cos q_2 \sin q_3 = L \cos\theta \sin\phi \qquad y = q_1 \sin q_2 \sin q_3 = L \sin\theta \sin\phi$$

$$z = -q_1 \cos q_3 = -L \cos\phi \qquad \textbf{[4.2.3]}$$

If the length of the pendulum is constant, $q_1 = L =$ constant, we do not need to use it as a variable; $q_2 = \theta$ and $q_3 = \phi$ are sufficient. If we use the coordinates x, y, and z to describe the motion, we have to relate them employing the *constraint relation*

$$x^2 + y^2 + z^2 = L^2 = \text{constant} \qquad \textbf{[4.2.4]}$$

Constraint relations, such as the one in this equation, indicate that the generalized coordinates are related to each other, and that the system can be analyzed by a smaller number of coordinates. The double link in Fig. 4.3, where the lengths of the rods are constant, requires at least two generalized coordinates to describe the configuration of the two rods. One can conveniently select them as the angles θ_1 and θ_2.

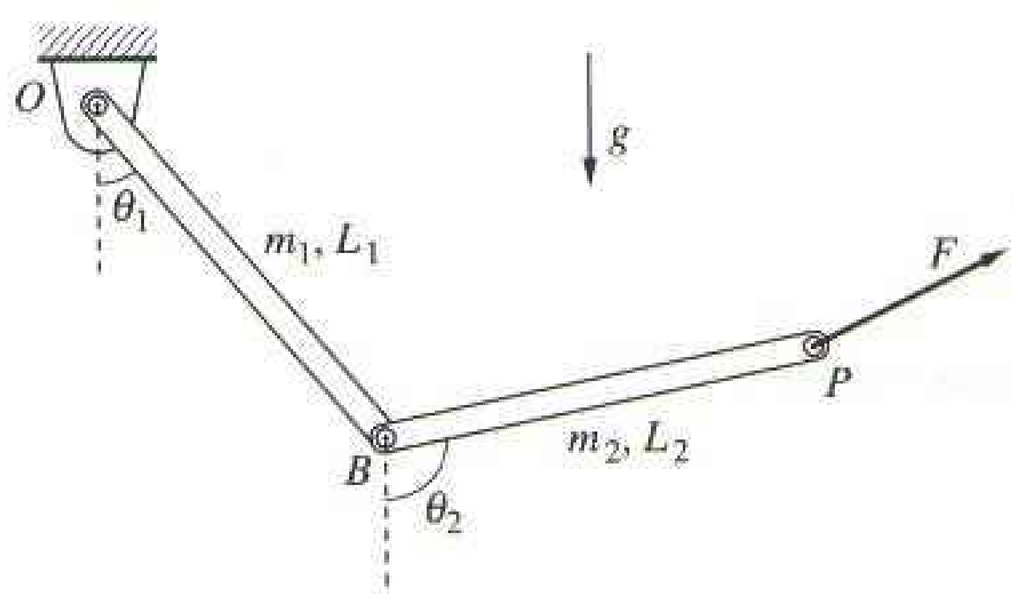

Figure 4.3 A double link

We hence need to distinguish between sets of generalized coordinates where each coordinate is independent of the others and where these variables are not independent.[2] In general, if a system of N particles has m constraint equations acting on it, we can describe the system uniquely by p *independent generalized coordinates* q_k, $(k = 1, 2, \ldots, p)$, where

$$p = 3N - m = n - m \quad \textbf{[4.2.5]}$$

in which p is called the number of degrees of freedom of the system. The term *degree of freedom* can be defined as the minimum number of *independent* coordinates necessary to describe a system *uniquely*. Sets of generalized coordinates where each coordinate is not independent of the others are called *constrained generalized coordinates* or *dependent generalized coordinates*. The number of degrees of freedom is a characteristic of the dynamical system and is independent of the coordinates used to describe the motion. While one can select the number and types of generalized coordinates and associated constraints in more than one way, $p = n - m$ is invariant.

The rate of change of a generalized coordinate with respect to time is called the *generalized velocity* and is denoted by $\dot{q}_k(t)$ $(k = 1, 2, \ldots, n)$. The $2n$-dimensional space spanned by the generalized coordinates and generalized velocities is called the *state space*.

For the pendulum in Fig. 4.2 we generated two sets of generalized coordinates. We could select other sets of generalized coordinates as well. For example, we could select the generalized coordinates as L, ϕ, and x. However, this would introduce some ambiguity into the description of the pendulum, as x has the same value when the angle θ is positive or negative. Such coordinates are known as *ambiguous generalized coordinates*. Another example of ambiguous generalized coordinates would be to use the coordinates x_P and y_P of the endpoint P of the double link in Fig. 4.3. One can easily show that a given coordinate of the endpoint can be reached by two different configurations of the links, the two being mirror images about the line joining points O and P, as shown in Fig. 4.4.

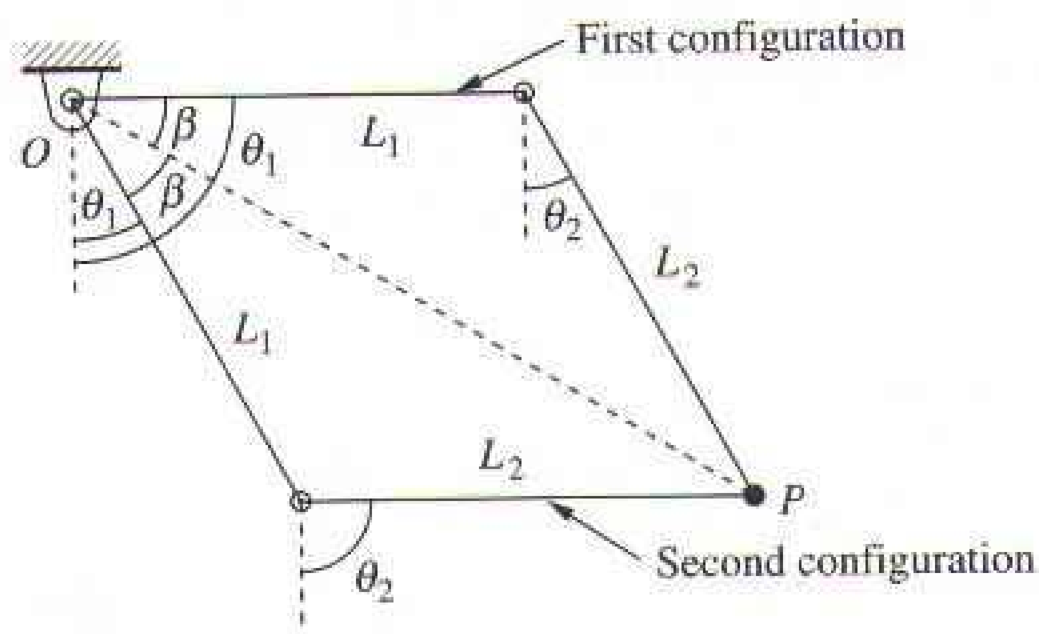

Figure 4.4

[2]In this regard, the definition of *generalized coordinate* here is slightly different than the traditional definition in older texts, which often restrict the term's meaning to only an independent set.

We draw two conclusions from the above. First, the generalized coordinates, whether they are independent or not, do not constitute a unique set. This actually is a tremendous advantage, as it gives a lot of flexibility. Second, one must exercise care *when selecting generalized coordinates, especially independent generalized coordinates, to avoid redundancies and ambiguities. A poor choice of generalized coordinates can make the problem formulation and solution unnecessarily difficult.*

The discussion here with regards to generalized coordinates is similar to the analysis of coordinate systems in Chapter 1. When we go from Cartesian to cylindrical or spherical coordinates, all we are doing is going from one set of generalized coordinates to another. We choose the coordinate system so that it simplifies the formulation.

4.3 CONSTRAINTS

In this section we analyze constraints that act on dynamical systems. We describe the constraints in terms of physical as well as generalized coordinates. The interest is primarily in equality constraints.

In dynamical systems, constraints are usually encountered as a result of contact between two (or more) bodies. Constraints restrict the motion of the bodies on which they act. Associated with a constraint are a *constraint equation* and a *constraint force*. The constraint equation describes the geometry and/or kinematics of the contact. The constraint force is the contact force, also called the *reaction*. (Constraint equations can also be written when the motion is viewed from a moving reference frame and there is no contact. The relative motion equation becomes the constraint equation.)

Consider Fig. 4.5 and a particle moving on a smooth surface whose shape is described by

$$f(x, y, z, t) = 0 \qquad \textbf{[4.3.1]}$$

where f has continuous second derivatives in all its variables. The motion of the particle over the surface can be viewed as the motion of an otherwise free particle subjected to the constraint of moving on that particular surface. Hence, $f(x, y, z, t) = 0$ represents a constraint equation. The constraint equation [4.3.1] is referred to as a *configuration constraint*. For a system described in terms of n generalized

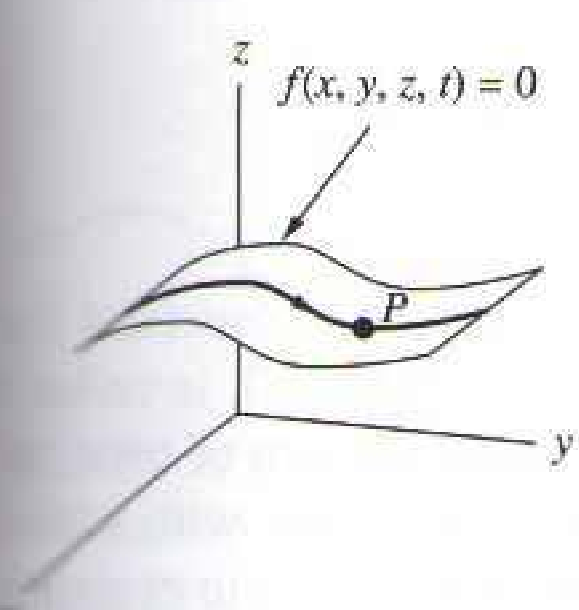

Figure 4.5 A particle moving on a smooth surface

coordinates, we can express a configuration constraint as

$$f(q_1, q_2, \ldots, q_n, t) = 0 \qquad [4.3.2]$$

The differential of the constraint f (in terms of physical and generalized coordinates) is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt = 0 \qquad [4.3.3a]$$

$$df = \frac{\partial f}{\partial q_1}dq_1 + \frac{\partial f}{\partial q_2}dq_2 + \cdots + \frac{\partial f}{\partial q_n}dq_n + \frac{\partial f}{\partial t}dt = 0 \qquad [4.3.3b]$$

The expressions [4.3.3] are said to be *constraint relations in Pfaffian form.* (A constraint in Pfaffian form is one that is expressed in the form of differentials.) Dividing these equations by dt, we write the *constraint equations in velocity form* (also called *velocity constraints* or *motion constraints*) as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t} = 0 \qquad [4.3.4a]$$

$$\frac{df}{dt} = \frac{\partial f}{\partial q_1}\dot{q}_1 + \frac{\partial f}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial f}{\partial q_n}\dot{q}_n + \frac{\partial f}{\partial t} = 0 \qquad [4.3.4b]$$

The general form of a velocity constraint can be written in terms of physical coordinates as

$$a_x\dot{x} + a_y\dot{y} + a_z\dot{z} + a_0 = 0 \qquad [4.3.5]$$

and, in terms of a system with n generalized coordinates subjected to m constraints,

$$\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0 \qquad j = 1, 2, \ldots, m \qquad [4.3.6]$$

where a_x, a_y, a_z, a_0, and a_{jk} and a_{j0} ($j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$) are functions of the generalized coordinates and time, for example, $a_{jk} = a_{jk}(q_1, q_2, \ldots, q_n, t)$. Note that once the constraints are imposed to a set of independent generalized coordinates, these coordinates are no longer independent.

A constraint that can be expressed as both a configuration constraint as well as in velocity form is called *holonomic.* Constraints that do not have this property are called *nonholonomic.* In other words, nonholonomic constraints cannot be expressed as configuration constraints.

4.3.1 HOLONOMIC CONSTRAINTS

An unconstrained dynamical system or one subjected to a holonomic constraint that is not an explicit function of time, for example, $f_j(q_1, q_2, \ldots, q_n) = 0$, is called a *scleronomic* system. If the holonomic constraint is an explicit function of time, the system is called *rheonomic.* Throughout this text we will deal mostly with scleronomic systems, as they constitute the majority of situations encountered in engineering applications.

Consider the single particle discussed above and the case when the holonomic constraint f is not an explicit function of time. That is, the plane defined by the constraint is fixed. Elimination of the $\partial f/\partial t$ term from Eq. [4.3.4a] yields

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} = 0 \qquad \textbf{[4.3.7]}$$

Denote the position and velocity of the particle by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ and $\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}$. The gradient of the constraint is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \qquad \textbf{[4.3.8]}$$

Taking the dot product between the gradient of the constraint and velocity $\mathbf{v}(t)$ gives

$$\nabla f \cdot \mathbf{v} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} \qquad \textbf{[4.3.9]}$$

which, when compared with Eq. [4.3.4a], yields

$$\nabla f \cdot \mathbf{v} = \frac{df}{dt} = 0 \qquad \textbf{[4.3.10]}$$

with the expected result that the particle velocity is always tangent to the surface.[3] The same relation can be derived for generalized coordinates.

Given the holonomic constraint of a particle moving on a surface, the question then arises as to what keeps the particle on the surface. The answer is a constraint force normal to the surface, as shown in Fig. 4.6. To every constraint relation corresponds a constraint force. Considering a single particle and denoting the constraint force by $\mathbf{F}'$, one can express it as

$$\mathbf{F}' = F'\mathbf{n} \qquad \textbf{[4.3.11]}$$

where $\mathbf{n}$ is a unit vector representing the direction perpendicular to the surface, usually referred to as the normal direction. (This direction is similar to the normal direction in normal-tangential coordinates, but here it can be taken as in either direction perpendicular to the surface.) Since $\mathbf{F}'$ is perpendicular to the surface, it must be perpendicular to the velocity. It follows from Eq. [4.3.9] that the unit vector $\mathbf{n}$, which is

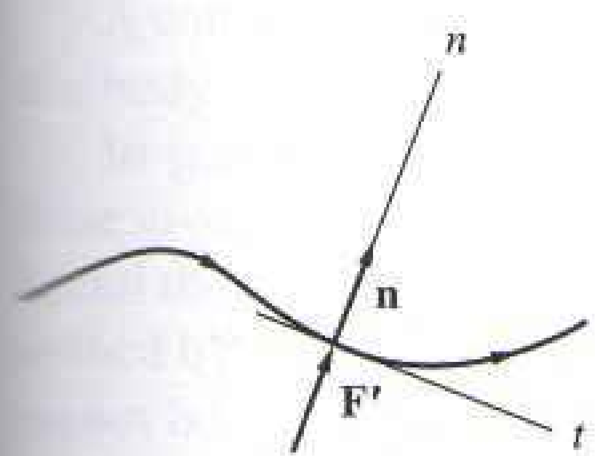

Figure 4.6 Constraint force for a holonomic constraint

[3] Recall the derivation in Chapter 1 when analyzing path variables that the particle velocity is always tangent to the path.

normal to the surface, should be parallel to ∇f. One can define $\mathbf{n}$ as

$$\mathbf{n} = \frac{\pm\nabla f}{|\nabla f|} = \pm\frac{\dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}}{\left[\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + \left(\dfrac{\partial f}{\partial z}\right)^2\right]^{1/2}} \qquad \textbf{[4.3.12]}$$

Since the constraint force is expressed as

$$\mathbf{F}' = F'_x\mathbf{i} + F'_y\mathbf{j} + F'_z\mathbf{k} \qquad \textbf{[4.3.13]}$$

when we compare Eqs. [4.3.12] and [4.3.13] we conclude that the components of the constraint force must be proportional to the partial derivatives of the constraint, or

$$\frac{F'_x}{\left(\dfrac{\partial f}{\partial x}\right)} = \frac{F'_y}{\left(\dfrac{\partial f}{\partial y}\right)} = \frac{F'_z}{\left(\dfrac{\partial f}{\partial z}\right)} \qquad \textbf{[4.3.14]}$$

Now, consider the work done by the constraint force as the particle moves from position $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$. Denoting this incremental work by dW and considering Eqs. [4.3.11] and [4.3.12], we obtain

$$dW = \mathbf{F}' \cdot d\mathbf{r} = F'\mathbf{n} \cdot d\mathbf{r} = \frac{F'}{|\nabla f|}\nabla f \cdot d\mathbf{r} = 0 \qquad \textbf{[4.3.15]}$$

This relation indicates that *the work done by a holonomic constraint force which is independent of time in any possible displacement is zero.* Such constraints are referred to as *workless constraints.* This result is to be expected, because the constraint force is always perpendicular to the velocity.

Note that, while the total work done by the constraint forces that are independent of time is zero, the individual constraint forces are doing work themselves. This work is in the form of transferring energy from one component of the system to the other. The sum of the transferred energies is zero. To visualize this, consider the double link in Fig. 4.3, whose free-body diagram is given in Fig. 4.7. If the first link is given an initial motion, the second link will begin moving, and vice versa. The motion of the second link is initiated by the constraint forces acting at point B.

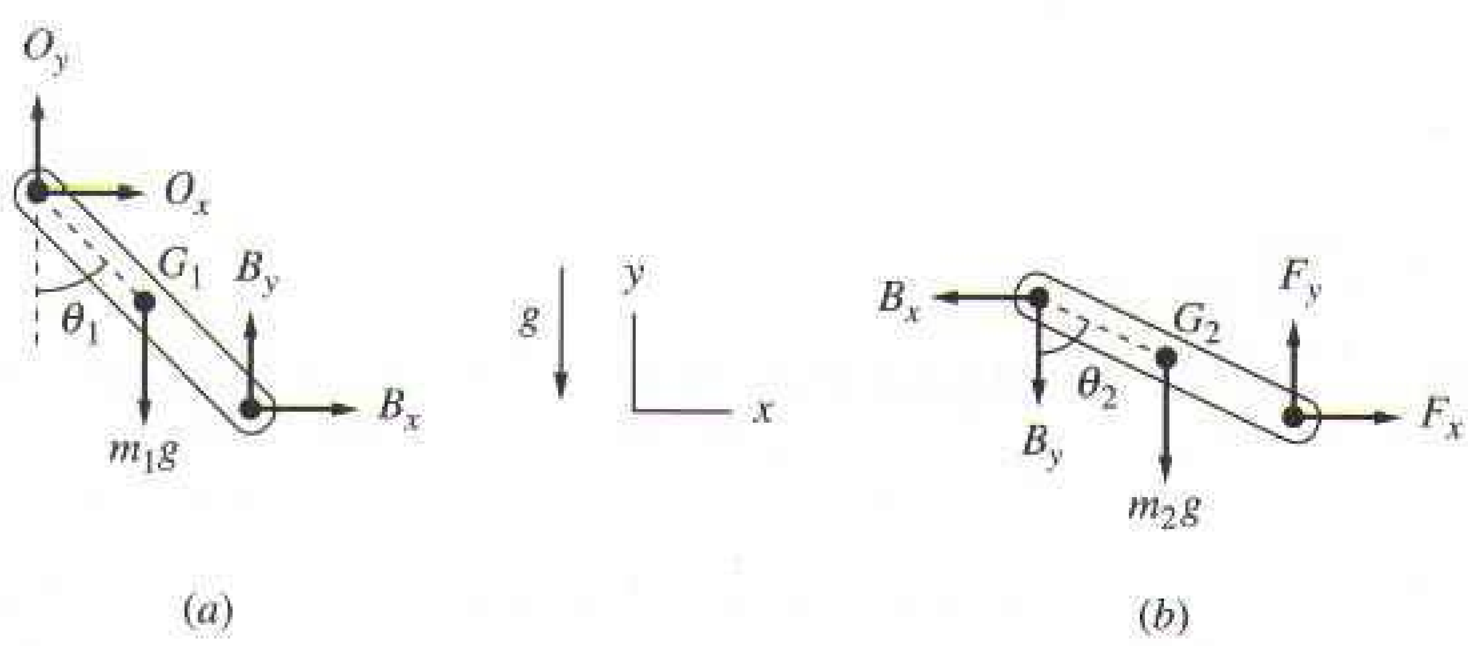

Figure 4.7 Free-body diagram of double link

Considering Fig. 4.7, reaction forces, such as the forces at the pin at O and at point B, are holonomic constraint forces. Normal forces are also holonomic constraint forces. However, friction forces are not constraint forces, even though their magnitude is directly dependent on a constraint force. Nevertheless, for static problems one can treat friction as a reaction force, because in such cases friction prevents motion.

Next consider a holonomic constraint that is an explicit function of time. For the particle considered earlier, this implies that the surface is moving and the constraint is in the form $f = f(x, y, z, t)$. Using Eqs. [4.3.4a] and [4.3.9] we obtain

$$\nabla f \cdot \mathbf{v} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} = -\frac{\partial f}{\partial t} \qquad \textbf{[4.3.16]}$$

which implies that $\nabla f \cdot d\mathbf{r} \neq 0$. It follows that the incremental work dW, which now is *not* a perfect differential as time is explicitly involved, is not zero. The incremental work has the form

$$dW = \mathbf{F}' \cdot d\mathbf{r} = F'\mathbf{n} \cdot d\mathbf{r} = \frac{F'}{|\nabla f|}\nabla f \cdot d\mathbf{r} \neq 0 \qquad \textbf{[4.3.17]}$$

When the holonomic constraint is time dependent, the work performed by the corresponding constraint force is not zero. The path followed by the particle can no longer be described by the path variables associated with the surface. The vector $\mathbf{n}$ describes the normal to the surface, but it is not the normal to the path followed by the particle.

4.3.2 NONHOLONOMIC CONSTRAINTS

When the constraint is nonholonomic, it can only be expressed in the form of Eqs. [4.3.5] or [4.3.6], as an integrating factor does not exist to permit expression in the form of Eqs. [4.3.1] or [4.3.2]. Consequently, none of the preceding results we obtained regarding the work done by the constraint force are valid for nonholonomic constraints. The constraint force associated with a nonholonomic constraint cannot be expressed as a force normal to a surface, as the nonholonomic constraint does not define a surface. One can go into the space spanned by $q_i(t)$ and $\dot{q}_i(t)$ $(i = 1, 2, \ldots, n)$ and define a surface there, but this does not give any physical insight or significant results. Hence, there is no general expression for the constraint force when the constraint is nonholonomic.

A common example of a nonholonomic constraint is the rolling without slipping of a body with no sharp corners or edges, such as a disk or a sphere.

In general, constraint equations in terms of relative velocities and especially those involving angular velocities that are not "simple" turn out to be nonholonomic. Recall the discussion of angular velocity in Chapter 2. When a reference frame is described by successive rotations about nonparallel axes, the resulting angular velocity cannot be described as the derivative of a vector.

Other examples of nonholonomic systems are from vehicle dynamics. Included in this category are the motions of ships, missiles, airplanes, automobiles, wheelbarrows, shopping carts, and sleds. Figure 4.8 is a simplified illustration of such a vehicle undergoing plane motion, such as a sled. Vehicles usually have a plane

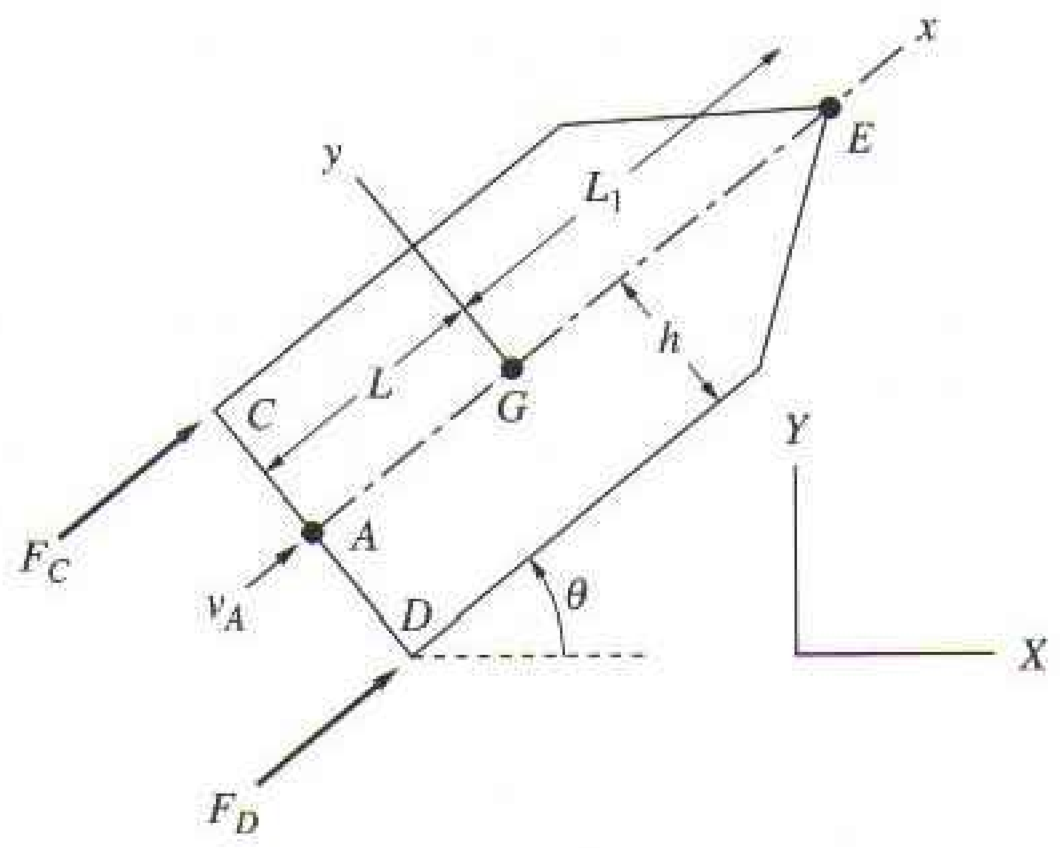

Figure 4.8 Generic model of a vehicle

of symmetry, and they are propelled in a way that the guiding forces act primarily along the symmetry plane, with a very small component of the force used to change direction. A steering mechanism usually accomplishes the change in direction.

One then makes the assumption that there is a point along the plane of symmetry, denoted by A, such that the velocity of point A is always along the plane of symmetry. The location of this point depends on the vehicle and the types of forces that prevent point A from having a velocity component perpendicular to the plane of symmetry. In a tricycle or automobile, the point A is in the middle between the rear wheels. In a boat, the hydrodynamic forces determine the location of A.

Consider the vehicle in Fig. 4.8. The configuration of this system can be described by the coordinates of point A, X_A and Y_A, and by the angle the body makes with the inertial X axis, denoted by θ. The nonholonomic constraint is associated with the translational velocity of point A. Denoting this velocity by $\mathbf{v}_A$, we write it as

$$\mathbf{v}_A = \dot{X}_A\mathbf{I} + \dot{Y}_A\mathbf{J} \qquad [4.3.18]$$

The constraint is written as

$$\mathbf{v}_A \cdot \mathbf{j} = 0 \qquad [4.3.19]$$

where $\mathbf{j} = \cos\theta\mathbf{J} - \sin\theta\mathbf{I}$. Introducing Eq. [4.3.19] into Eq. [4.3.18], we obtain

$$\begin{aligned}\mathbf{v}_A \cdot \mathbf{j} &= (\dot{X}_A\mathbf{I} + \dot{Y}_A\mathbf{J}) \cdot (-\sin\theta\mathbf{I} + \cos\theta\mathbf{J}) \\ &= -\dot{X}_A \sin\theta + \dot{Y}_A \cos\theta = 0\end{aligned} \qquad [4.3.20]$$

This equation can conveniently be expressed as

$$\dot{X}_A - \frac{\dot{Y}_A}{\tan\theta} = 0 \qquad [4.3.21]$$

It is clear that this constraint is nonholonomic. The associated constraint force is basically the resistance of point A to have any motion perpendicular to the line of motion. In an automobile, for example, this force would be the friction force between the rear tires and the road surface in the direction perpendicular to the velocity of the tires. A very strong wind in the lateral direction, collision with another vehicle, or taking a turn with high speed would violate this constraint.

In general, the constraint force associated with a nonholonomic constraint performs work. A special case when this is not valid is rolling without slipping, where the friction force is applied to a point with zero velocity. For roll without slip, friction becomes a constraint force, as it reduces the number of degrees of freedom.

We next look into determining whether a constraint is holonomic or not. In general, whether it is or is not can be ascertained by visual inspection. Mathematically, in order for a constraint in Pfaffian or velocity form to be integrable to configuration form, the constraint relation must satisfy differentiability conditions. The constraint must represent an exact differential. Consider Eq. [4.3.6]. If the jth constraint equation is holonomic, one should be able to write it as $f_j(q_1, q_2, \ldots, q_n, t) = 0$. Taking the differential of f_j and, for the most general case, dividing it by an integrating factor $g_j(q_1, q_2, \ldots, q_n)$ we obtain Eq. [4.3.3b]. Comparing Eq. [4.3.4b] with Eq. [4.3.6], we obtain for the general case of a holonomic constraint

$$\frac{\partial f_j}{\partial q_k} = g_j a_{jk} \qquad \frac{\partial f_j}{\partial t} = g_j a_{j0} \qquad k = 1, 2, \ldots, n \tag{4.3.22}$$

For a constraint given by Eq. [4.3.6] to be holonomic, there must be a function f_j and an integrating factor $g_j(q_1, q_2, \ldots, q_n)$ $(j = 1, 2, \ldots, m)$ where the partial derivatives of $f_j (j = 1, 2, \ldots, m)$ satisfy Eq. [4.3.22]. To check this, we evaluate the second derivatives of f_j. Indeed, considering an index r, we obtain

$$\frac{\partial^2 f_j}{\partial q_k \partial q_r} = \frac{\partial}{\partial q_r}(g_j a_{jk}) \quad \text{and} \quad \frac{\partial^2 f_j}{\partial q_k \partial q_r} = \frac{\partial}{\partial q_k}(g_j a_{jr}) \tag{4.3.23}$$

$$\frac{\partial^2 f_j}{\partial q_r \partial t} = \frac{\partial}{\partial q_r}(g_j a_{j0}) \quad \text{and} \quad \frac{\partial^2 f_j}{\partial q_r \partial t} = \frac{\partial}{\partial t}(g_j a_{jr})$$
$$k, r = 1, \ldots, n;\ j = 1, 2, \ldots, m \tag{4.3.24}$$

From Eqs. [4.3.23] and [4.3.24] if an integrating factor g_j exists such that a_{jk} and a_{j0} satisfy the relations

$$\frac{\partial}{\partial q_r}(g_j a_{jk}) = \frac{\partial}{\partial q_k}(g_j a_{jr}) \qquad \frac{\partial}{\partial q_r}(g_j a_{j0}) = \frac{\partial}{\partial t}(g_j a_{jr})$$
$$r = 1, \ldots, n;\ j = 1, 2, \ldots, m \tag{4.3.25}$$

then the constraint is holonomic. The problem with using the above procedure is that it may not be easy to find the integrating factor, especially for systems having more than three degrees of freedom.

A constraint of the form $f(q_1, q_2, \ldots, q_n, t) \geq 0$, or $\sum a_k \dot{q}_k + a_0 \geq 0$, that is, an inequality constraint, is nonholonomic because it cannot be reduced to a form $f(q_1, q_2, \ldots, q_n, t) = 0$. Such constraints require a different treatment than equality constraints. We also encounter constraints that are valid in some positions of the body or during certain intervals of the motion. Such constraints can also be classified as inequality constraints. They can be found in problems involving contact.

Consider now a system that originally has n degrees of freedom and is subjected to m holonomic constraints. Introduction of m constraints reduces the degrees of freedom by m to $p = n - m$, resulting in a set of m excess, or surplus, coordinates.

It is possible, at least mathematically, to eliminate the surplus coordinates from the formulation, which results in an unconstrained system of order $n - m$. Because of this, unconstrained systems are referred to as holonomic.

By contrast, a nonholonomic constraint constrains only the generalized velocities, without affecting the generalized coordinates. In such systems there are n independent generalized coordinates and $n - m$ independent generalized velocities.

Example 4.1

A bead is sliding in a tube, whose shape is given by the equation $y = 1 - x^2$, as shown in Fig. 4.9. Find the direction of the normal to the tube.

Solution

One can solve this problem in a number of ways. We first consider the problem from a physical standpoint. Because the bead is sliding in the tube, the equation defining the shape of the tube becomes the constraint equation, and it has the form

$$f(x, y) = y - 1 + x^2 = 0 \qquad \textbf{[a]}$$

Taking the partial derivatives of f, we obtain

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 1 \qquad \textbf{[b]}$$

so that, using Eq. [4.3.8], the gradient of f has the form $\nabla f = 2x\mathbf{i} + \mathbf{j}$. From Eq. [4.3.12], the unit vector in the normal direction (chosen, for convenience, positive outward) becomes

$$\mathbf{n} = \frac{2x\mathbf{i} + \mathbf{j}}{\sqrt{1 + 4x^2}} \qquad \textbf{[c]}$$

As expected, because the constraint is not an explicit function of time, neither is the direction of the constraint force. The constraint is, of course, holonomic.

To solve this problem geometrically, we define the angle θ between the horizontal and the tangent to the curve. The tangent of $\theta(x)$ describes the slope of the tube, and

$$\tan\theta = \frac{dy}{dx} = -2x \qquad \textbf{[d]}$$

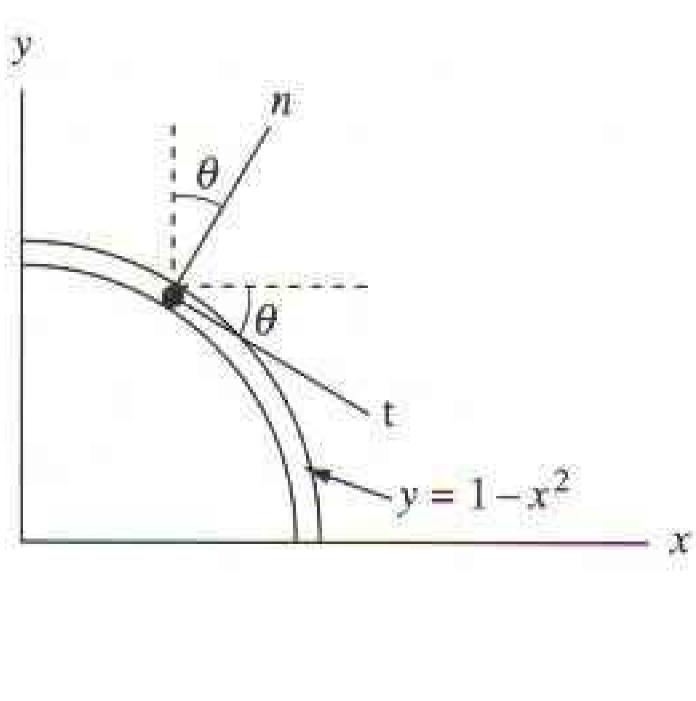

Figure 4.9 Bead sliding inside a tube

We find the sine and cosine of θ by

$$\sin\theta = \frac{2x}{\sqrt{1+4x^2}} \qquad \cos\theta = -\frac{1}{\sqrt{1+4x^2}} \qquad \textbf{[e]}$$

Now we can express the unit vector describing the normal as

$$\mathbf{n} = \sin\theta\mathbf{i} - \cos\theta\mathbf{j} = \frac{2x\mathbf{i}+\mathbf{j}}{\sqrt{1+4x^2}} \qquad \textbf{[f]}$$

We can also determine the normal direction directly from the geometry, using the approach in Chapter 1, without going into any constraint equations. Denoting the path variable by x, we write the position vector as

$$\mathbf{r} = x\mathbf{i} + (1-x^2)\mathbf{j} \qquad \textbf{[g]}$$

and the expressions for the slope and the unit vector in the tangential direction become

$$\mathbf{r}' = \frac{d\mathbf{r}}{dx} = \mathbf{i} - 2x\mathbf{j} \qquad \mathbf{e}_t = \frac{\mathbf{i}-2x\mathbf{j}}{\sqrt{1+4x^2}} \qquad s' = \sqrt{1+4x^2} \qquad \textbf{[h]}$$

Use of Eq. [1.3.36] yields $\mathbf{n}$. When the path parameters associated with the motion of a body are specified, in essence a constraint has been imposed on an otherwise free body.

Example 4.2

A block of mass m is attached to a cord of original length L and is rotating about a thin hub, as shown in Fig. 4.10. Friction is negligible. Find the constraint force if (*a*) the cord is not wrapping around the hub, and (*b*) the cord is wrapping around the hub.

Solution

a. When the cord is not wrapping around the hub, the constraint is holonomic and independent of time. The constraint equation basically describes that the length of the cord is constant, and it has the form

$$f(x, y) = x^2 + y^2 - L^2 = 0 \qquad \textbf{[a]}$$

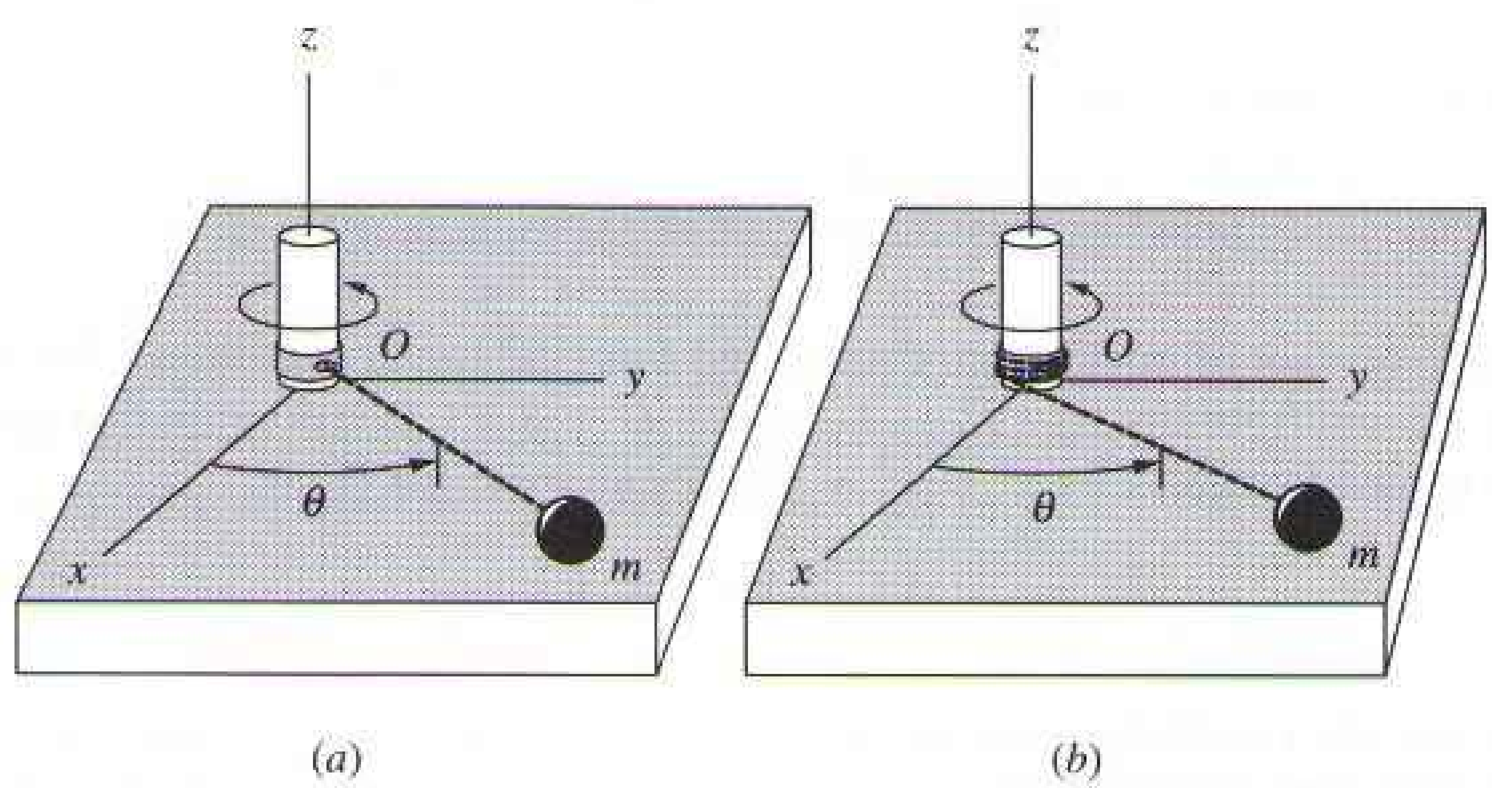

Figure 4.10 Mass rotating around a thin hub (*a*) Cord is not wrapping around hub (*b*) Cord is wrapping around hub

Once motion is initiated, the mass keeps rotating with the same speed and the energy of the particle does not change. The constraint force is the tension in the rope, and it does no work.

b. The situation is quite different when the rope wraps around the hub. Assuming that the hub radius is very small, the tension in the rope is directed toward point O. Summing moments about O, we obtain

$$\sum M_O = 0 \tag{b}$$

so that the angular momentum about O is conserved. In essence, we have a central force problem. Let us use polar coordinates r and θ. Consider that the length of the rope, denoted by r, reduces continuously by the relation

$$r = L - r_o\theta \tag{c}$$

where r_o is the radius of the hub. In one revolution of the mass, the rope shortens by $2\pi r_o$.

The angular momentum about O is given by $H_O = mr^2\dot{\theta}$. Because the angular momentum is conserved,

$$r^2\dot{\theta} = \text{constant} = h \tag{d}$$

where we note that the constant h is always greater than zero, $h > 0$, and that h is a function of the initial condition. Differentiating the relation between r and θ, we write

$$\dot{r} = -r_o\dot{\theta} \rightarrow \dot{\theta} = -\frac{\dot{r}}{r_o} \tag{e}$$

and substituting the above relation into Eq. [d], we obtain

$$r^2\dot{\theta} = \frac{-r^2\dot{r}}{r_o} = h \tag{f}$$

or

$$r^2\dot{r} = -r_o r^2\dot{\theta} = -r_o h = C \qquad C < 0 \tag{g}$$

where C is constant. Now, let us find the response $r(t)$. We can rewrite Eq. [g] as

$$r^2\,dr = C\,dt \tag{h}$$

which, when integrated, gives

$$\frac{r^3}{3} = Ct + D \tag{i}$$

where D is a constant of integration, determined from the initial conditions. We note that at $t = 0$, $r = L$, and from Eq. [i] $r^3/3 = D$, so that $D = L^3/3$. Considering that the length of the rope is related to x and y by $r^2 = x^2 + y^2$, we can write Eq. [i] as

$$f(x, y, t) = \frac{(x^2 + y^2)^{3/2}}{3} - \frac{L^3}{3} - Ct = 0 \tag{j}$$

The constraint is a time-dependent holonomic constraint, that is, a rheonomic constraint. The constraint force, which is the tension in the rope, does perform work. To show that the constraint force does indeed perform work, we consider the configuration vector $\mathbf{r}$ and its derivative

$$\mathbf{r} = r\mathbf{e}_r \qquad \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{k}$$

The constraint force (the tension in the rope) can be expressed as $\mathbf{F}' = -F\mathbf{e}_r$, so that the dot product between the constraint force and the particle velocity becomes

$$\mathbf{F}' \cdot \dot{\mathbf{r}} = -F\dot{r} \quad \textbf{[l]}$$

which is not zero. Note that for the case when the length of the rope is not changing, $\dot{r} = 0$, and the work done by the constraint is zero. Also note that in order to find an explicit expression for $r(t)$, the initial angular velocity must be specified.

Example 4.3

Given a system with generalized coordinates q_1 and q_2 and the constraint equation

$$\left(3q_1 \sin q_2 + \frac{q_2^2}{q_1} + 2\right) dq_1 + (q_1^2 \cos q_2 + 2q_2)\, dq_2 = 0$$

determine whether the constraint is holonomic or not.

Solution

The constraint equation is holonomic if there exists an integrating factor $g(q_1, q_2)$, such that Eq. [4.3.22] holds, or

$$\frac{\partial f}{\partial q_1} = 3gq_1 \sin q_2 + \frac{gq_2^2}{q_1} + 2g \qquad \frac{\partial f}{\partial q_2} = gq_1^2 \cos q_2 + 2gq_2 \quad \textbf{[a]}$$

We observe that if $g(q_1, q_2) = q_1$, then

$$\frac{\partial f}{\partial q_1} = 3q_1^2 \sin q_2 + q_2^2 + 2q_1 \qquad \frac{\partial f}{\partial q_2} = q_1^3 \cos q_2 + 2q_1 q_2 \quad \textbf{[b]}$$

Integrating the two expressions, we obtain

$$f = q_1^3 \sin q_2 + q_1 q_2^2 + q_1^2 + h_1(q_2) + C_1 \qquad f = q_1^3 \sin q_2 + q_1 q_2^2 + h_2(q_1) + C_2 \quad \textbf{[c]}$$

where h_1 and h_2 are functions that appear as a result of the integration over q_1 and q_2, respectively, and C_1 and C_2 are constants. Comparing the two integrated terms, we conclude that $h_2(q_1) = q_1^2$ and $h_1(q_2) = 0$ and that the constants are related by $C_1 = C_2$. The constraint, therefore, is holonomic and has the form

$$f(q_1, q_2) = q_1^3 \sin q_2 + q_1 q_2^2 + q_1^2 + C = 0 \quad \textbf{[d]}$$

where C is a constant. For this problem the integrating factor was found by visual inspection. In general, there are no set guidelines for finding the integrating factor.

Example 4.4

The tip of the double-link mechanism in Fig. 4.11 is constrained to lie on the inclined plane. Derive the constraint equation and express it in velocity form.

Solution

This is a single degree of freedom system. We use θ_1 and θ_2 as generalized coordinates. Hence, we need one constraint equation. We can simplify the formulation by expressing the position of the tip along the incline by the variable s. To derive the constraint equation we write the position vector of the tip in two ways: using the links and using the incline. Using the links, the position vector has the form

$$\mathbf{r}_P = (L_1 \cos\theta_1 + L_2 \cos\theta_2)\mathbf{i} + (L_1 \sin\theta_1 + L_2 \sin\theta_2)\mathbf{j} \quad \textbf{[a]}$$

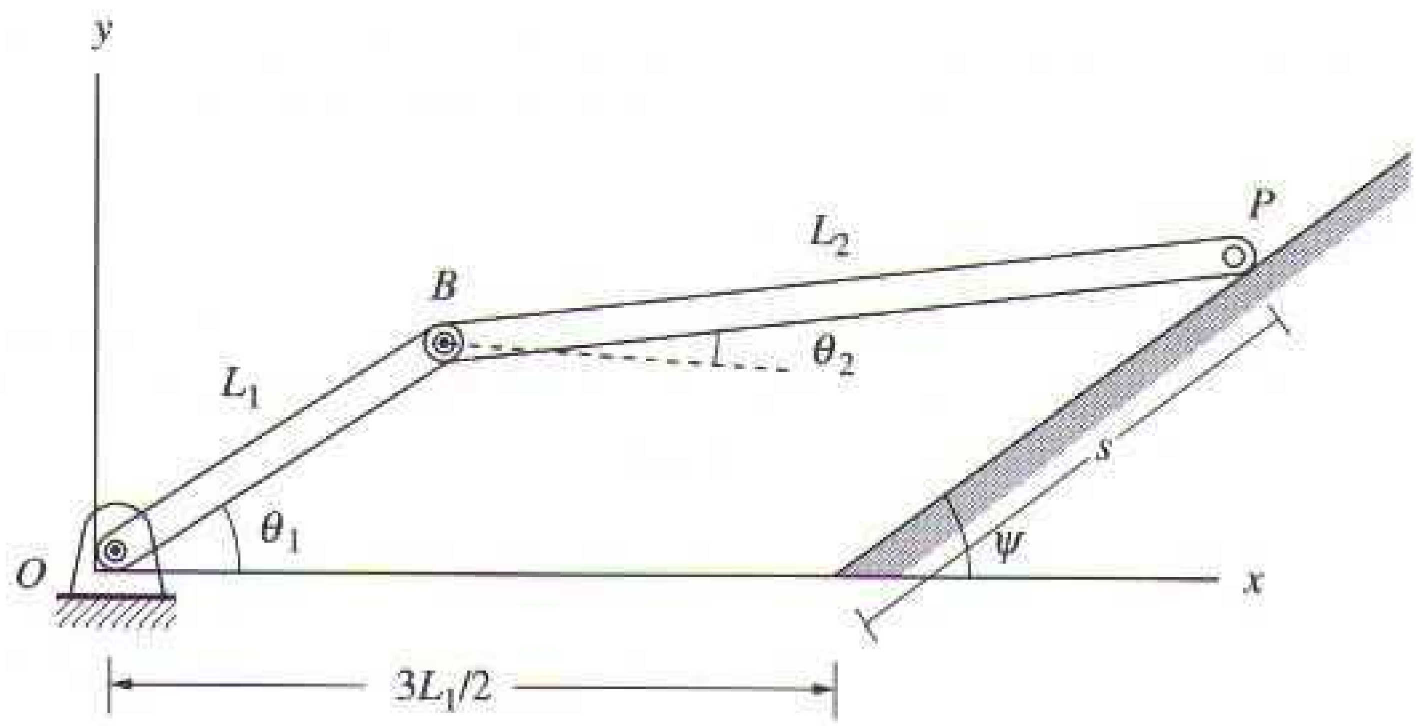

Figure 4.11

and using the incline, it has the form

$$\mathbf{r}_P = \frac{3L_1}{2}\mathbf{i} + s\cos\psi\mathbf{i} + s\sin\psi\mathbf{j} \quad \textbf{[b]}$$

We equate the above two expressions and separate components in the x and y directions, thus

$$L_1\cos\theta_1 + L_2\cos\theta_2 = \frac{3L_1}{2} + s\cos\psi \quad \textbf{[c]}$$

$$L_1\sin\theta_1 + L_2\sin\theta_2 = s\sin\psi \quad \textbf{[d]}$$

To obtain the constraint equation, we eliminate s by multiplying Eq. [c] by $\sin\psi$ and Eq. [d] by $-\cos\psi$ and adding the two equations. Dividing the result by $L_1\sin\psi$, we obtain

$$\cos\theta_1 + \frac{L_2}{L_1}\cos\theta_2 - \frac{1}{\tan\psi}\sin\theta_1 - \frac{L_2}{L_1\tan\psi}\sin\theta_2 = \frac{3}{2} \quad \textbf{[e]}$$

which is recognized as the holonomic constraint equation. To express this constraint in velocity form, we differentiate Eq. [e] with respect to time, with the result

$$\left(\sin\theta_1 + \frac{\cos\theta_1}{\tan\psi}\right)\dot{\theta}_1 + \frac{L_2}{L_1}\left(\sin\theta_2 + \frac{\cos\theta_2}{\tan\psi}\right)\dot{\theta}_2 = 0 \quad \textbf{[f]}$$

4.4 VIRTUAL DISPLACEMENTS AND VIRTUAL WORK

At this point, we introduce the variational notation. The variational notation is ideally suited for dynamics problems because it makes the formulation concise, and it has a meaningful physical interpretation. When applied to dynamical systems, the variations of displacements are known as *virtual displacements,* denoted by δx, δy, δz, etc. In terms of generalized coordinates, the virtual displacements have the form $\delta q_1, \delta q_2, \ldots, \delta q_n$. The variations of the velocities are denoted by $\delta\dot{x}$, $\delta\dot{y}$, $\delta\dot{z}$ for physical coordinates and $\delta\dot{q}_k$ $(k = 1, 2, \ldots, n)$ for generalized velocities.

Virtual displacements have the following properties:

- They are infinitesimal displacements.
- They are consistent with the system constraints, but are arbitrary otherwise.
- The variation of displacements (or velocities, etc.) is obtained by holding time fixed; therefore, virtual displacements can be considered as occurring instantaneously, and time is not involved in their applications.

Dealing with virtual displacements is like imagining the system in a different position that is physically realizable, while freezing time. It is as if a different set of forces were applied and, as a result, the system moved to another location by one of the admissible paths it can follow.

The rules for calculating virtual displacements are intimately related to the rules of differentiation. For the position vector $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, or $\mathbf{r} = \mathbf{r}(q_1, q_2, \ldots, q_n, t)$, the variation of $\mathbf{r}$ becomes

$$\delta\mathbf{r} = \delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k} \quad \text{or} \quad \delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial q_1}\delta q_1 + \frac{\partial\mathbf{r}}{\partial q_2}\delta q_2 + \ldots + \frac{\partial\mathbf{r}}{\partial q_n}\delta q_n \tag{4.4.1a,b}$$

Figure 4.12 depicts the concept of a variation (for the coordinate y). When expressing the motion $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in which x, y, and z are all functions of the generalized coordinates, the variation of $\mathbf{r}$ has the form

$$\delta\mathbf{r} = \sum_{k=1}^{n}\left(\frac{\partial x}{\partial q_k}\mathbf{i} + \frac{\partial y}{\partial q_k}\mathbf{j} + \frac{\partial z}{\partial q_k}\mathbf{k}\right)\delta q_k \tag{4.4.2}$$

As discussed in Appendix B, we distinguish between dependent and independent variables. For dynamical systems, time is the independent variable. The coordinates x, y, z, as well $q_1, q_2, \ldots, q_n$ are functions of time and are referred to as the dependent variables. The term *dependent* is used here to denote explicit dependence of the generalized coordinates on time, rather than on each other. It follows that one can interchange the time differentiation and the variation operators. That is, $\delta\dot{q}_k = \delta(dq_k/dt) = d(\delta q_k)/dt$ $(k = 1, 2, \ldots, n)$.

The variation of a position vector can be obtained in two different ways. One way is by obtaining an analytical expression for the position vector and taking its variation

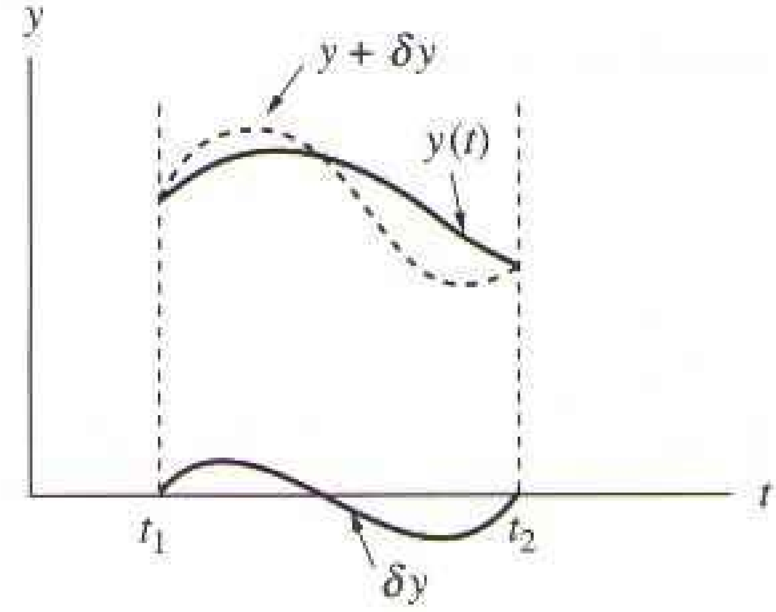

Figure 4.12 Variation of y

by differentiating with respect to the generalized coordinates. Basically this is the use of Eq. [4.4.1b]; it is known as the *analytical approach.* This approach may lead to lengthy expressions for certain complex problems. When $\mathbf{r}$ is expressed in terms of the coordinates of a moving reference frame, one must also take the variation of the unit vectors of the moving reference frame. The exception to this is when the motion of the relative frame is prespecified as a known quantity and is not treated as a motion variable.

In the second way, known as the *kinematical approach,* one explores similarities between velocities and virtual displacements. When taking the variation of an expression, the independent variable is not varied. We use this property, as time is the independent variable. The time derivative of $\mathbf{r}$ is

$$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} + \frac{\partial \mathbf{r}}{\partial t} \qquad \dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{r}}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \mathbf{r}}{\partial q_n}\dot{q}_n + \frac{\partial \mathbf{r}}{\partial t} \tag{4.4.3a,b}$$

Elimination of the partial derivative of $\mathbf{r}$ with respect to time, elimination of all expressions explicit in time, and replacement of $\dot{x}$ by δx, $\dot{y}$ by δy, $\dot{z}$ by δz in Eq. [4.4.3a] and of $\dot{q}_k$ $(k = 1, 2, \ldots, n)$ with δq_k in Eq. [4.4.3b] yields the variation of $\mathbf{r}$. This implies that if the expression for the velocity is known, the associated virtual displacement can be obtained directly from it. This approach of calculating virtual displacements from velocities is especially useful when the velocity of a point can be found using an instant center or a relative velocity expression, such as

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} + \mathbf{v}_{B\text{rel}} \tag{4.4.4}$$

The variation of the displacement of point B is

$$\delta\mathbf{r}_B = \delta\mathbf{r}_A + \boldsymbol{\delta\theta} \times \mathbf{r}_{B/A} + \delta\mathbf{r}_{B\text{rel}} \tag{4.4.5}$$

where we note that the $\mathbf{r}_{B/A}$ term is left intact and that $\boldsymbol{\delta\theta}$ represents the variation of an infinitesimal rotation. Also, keeping in line with the developments in Chapter 2, we extend the boldface to the entire term $\boldsymbol{\delta\theta}$ to denote that $\boldsymbol{\delta\theta}$ is a variation of a rotation and that it is not obtained by differentiating a vector.

Consider Eq. [4.4.3b] and the derivative of $\dot{\mathbf{r}}$ with respect to $\dot{q}_k$. Of all the terms in Eq. [4.4.3b] only one survives and we obtain the important relationship

$$\frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}}{\partial q_k} \qquad k = 1, 2, \ldots, n \tag{4.4.6}$$

so that the variation of $\mathbf{r}$ can be expressed as

$$\delta\mathbf{r} = \sum_{k=1}^{n} \frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_k}\delta q_k \tag{4.4.7}$$

Note that Eq. [4.4.7] is in essence the mathematical representation of the kinematical method of calculating virtual displacements. Next, consider the holonomic constraint $f(x, y, z, t) = 0$ and obtain its variation, which has the form

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = 0 \tag{4.4.8}$$

Because time is held fixed while f is varied, δf has the same form whether the constraint is time dependent or not. When a constraint is given in velocity form by Eqs. [4.3.5] and [4.3.6], in terms of physical coordinates the virtual displacements satisfy

$$a_x\delta x + a_y\delta y + a_z\delta z = 0 \qquad \textbf{[4.4.9]}$$

and, in terms of generalized coordinates and the jth constraint, they satisfy

$$\delta f_j = a_{j1}\delta q_1 + a_{j2}\delta q_2 + \cdots + a_{jn}\delta q_n \qquad j = 1, 2, \ldots, m \qquad \textbf{[4.4.10]}$$

Let us next consider the work done by a force over a virtual displacement. Consider a body acted upon by a force $\mathbf{F}$ and the virtual displacement associated with the point at which the force $\mathbf{F}$ is applied. We define the work done by the force over the virtual displacement $\delta\mathbf{r}$ as the *virtual work* or *variation of work* and denote it by δW. Hence

$$\delta W = \mathbf{F}\cdot\delta\mathbf{r} \qquad \textbf{[4.4.11]}$$

We will examine the virtual work associated with a general force in the next section. For now, let us consider the holonomic constraint $f(x, y, z, t) = 0$ and the associated virtual work. Recall that whether the constraint is time dependent or not is immaterial. From Eqs. [4.3.11]–[4.3.14], the constraint force $\mathbf{F}'$ has the form

$$\mathbf{F}' = F'_x\mathbf{i} + F'_y\mathbf{j} + F'_z\mathbf{k} = \frac{F'}{|\nabla f|}\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = \frac{F'}{|\nabla f|}\nabla f \qquad \textbf{[4.4.12]}$$

We define by $\delta W'$ the work performed by the constraint force in any virtual displacement *virtual work due to constraint forces,* as

$$\delta W' = \mathbf{F}'\cdot\delta\mathbf{r} = F'_x\delta x + F'_y\delta y + F'_z\delta z \qquad \textbf{[4.4.13]}$$

Using Eqs. [4.4.1] and [4.4.8] we conclude that

$$\delta W' = \mathbf{F}'\cdot\delta\mathbf{r} = \frac{F'}{|\nabla f|}\nabla f\cdot\delta\mathbf{r} = \frac{F'}{|\nabla f|}\delta f = 0 \qquad \textbf{[4.4.14]}$$

so that *the work performed by a holonomic constraint force in any virtual displacement is zero.*

Example 4.5

A disk of radius R rolls without slipping on a rod of length L pivoted at one end, as shown in Fig. 4.13. Denoting the pivot angle by θ and the angular displacement of the disk by ϕ, find the virtual displacement of the center of the disk.

Solution

We will solve this problem using both a kinematical and an analytical approach. We begin with the kinematical approach. We select an inertial frame XYZ and a relative frame xyz, such that the xyz axes are obtained by rotating the XYZ frame by an angle θ counterclockwise about the Z axis.

The velocity of point G can be written as

$$\mathbf{v}_G = \mathbf{v}_B + \boldsymbol{\omega}\times\mathbf{r}_{G/B} + \mathbf{v}_{\text{rel}} \qquad \textbf{[a]}$$

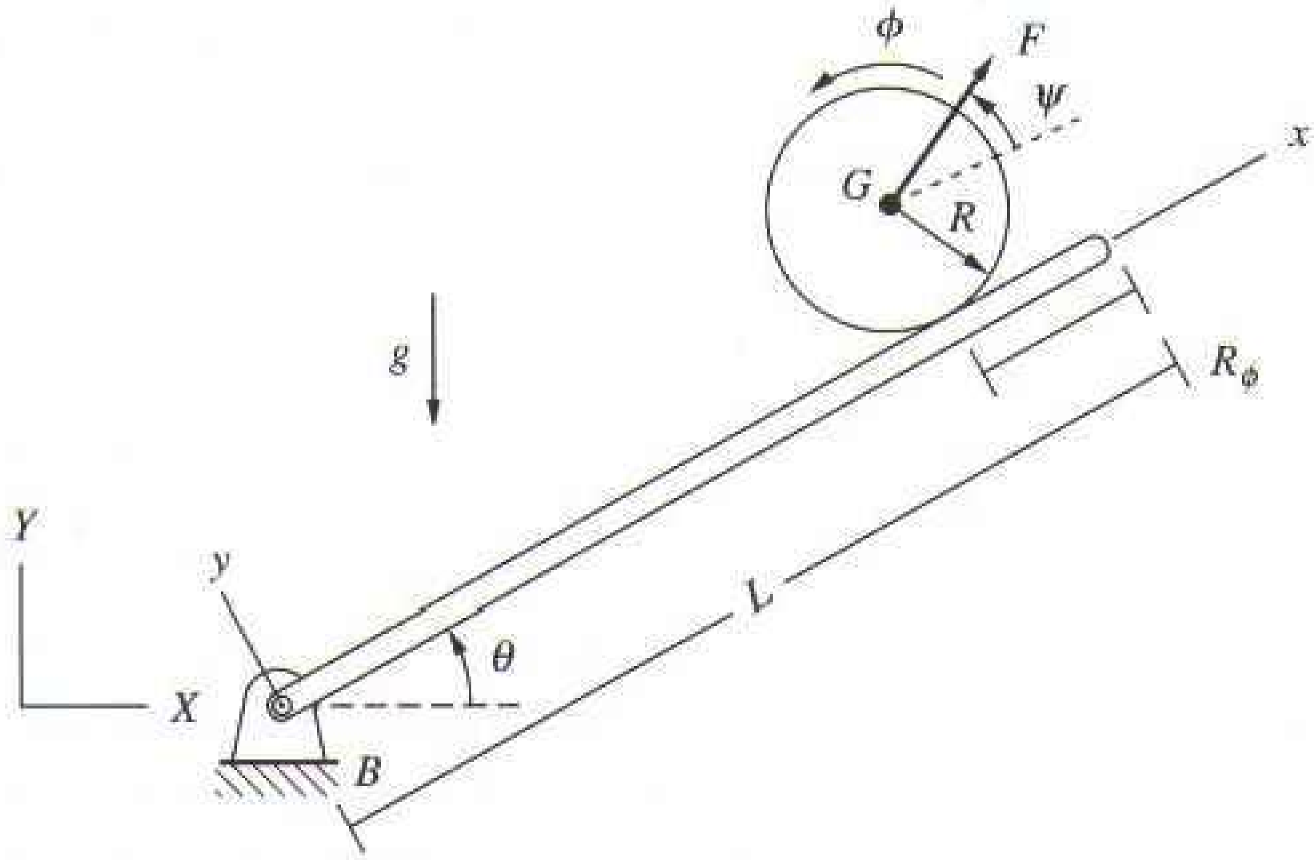

Figure 4.13 Disk rolling over bar

in which $\mathbf{v}_B = 0$, $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, and

$$\mathbf{r}_{G/B} = \mathbf{r}_G = (L - R\phi)\mathbf{i} + R\mathbf{j} \qquad \mathbf{v}_{\text{rel}} = -R\dot{\phi}\mathbf{i} \tag{b, c}$$

Substituting the above values into Eq. [a] we obtain

$$\mathbf{v}_G = \dot{\theta}\mathbf{k} \times [(L - R\phi)\mathbf{i} + R\mathbf{j}] - R\dot{\phi}\mathbf{i} = -R(\dot{\phi} + \dot{\theta})\mathbf{i} + (L\dot{\theta} - R\phi\dot{\theta})\mathbf{j} \tag{d}$$

Thus, we write the variation of $\mathbf{r}_G$ as

$$\delta\mathbf{r}_G = -R(\delta\phi + \delta\theta)\mathbf{i} + (L\,\delta\theta - R\phi\,\delta\theta)\mathbf{j} \tag{e}$$

Now we will find the variation of $\mathbf{r}_G$ analytically. The position vector $\mathbf{r}_G$ is given in Eq. [b]. There are two ways to obtain its variation. In the first, we express $\mathbf{r}_G$ in terms of the inertial coordinate frame and then differentiate. In the second, we take the variation of Eq. [b] directly, which requires the variation of the unit vectors **i** and **j** of the moving frame. The relation between the unit vectors of the inertial and relative frames is

$$\mathbf{k} = \mathbf{K} \qquad \begin{bmatrix}\mathbf{i}\\ \mathbf{j}\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \mathbf{J}\end{bmatrix} \tag{f}$$

Introducing this into Eq. [b], we obtain

$$\mathbf{r}_G = [(L - R\phi)\cos\theta - R\sin\theta]\mathbf{I} + [(L - R\phi)\sin\theta + R\cos\theta]\mathbf{J} \tag{g}$$

The virtual displacement then becomes

$$\begin{aligned}\delta\mathbf{r}_G = {} & [-(L - R\phi)\sin\theta\,\delta\theta - R\cos\theta\,\delta\phi - R\cos\theta\,\delta\theta]\mathbf{I}\\ & + [(L - R\phi)\cos\theta\,\delta\theta - R\sin\theta\,\delta\phi - R\sin\theta\,\delta\theta]\mathbf{J}\end{aligned} \tag{h}$$

To convert the virtual displacement in terms of the relative frame, we introduce the relationships

$$\mathbf{I} = \cos\theta\mathbf{i} - \sin\theta\mathbf{j} \qquad \mathbf{J} = \sin\theta\mathbf{i} + \cos\theta\mathbf{j} \tag{i}$$

into Eq. [h], which gives Eq. [e].

Next, take the variation of Eq. [b] directly, with the result

$$\delta \mathbf{r}_G = -R\,\delta\phi \mathbf{i} + (L - R\phi)\,\delta \mathbf{i} + R\,\delta \mathbf{j} \qquad \textbf{[j]}$$

From Eq. [f], the variations of the unit vectors have the form

$$\delta \mathbf{i} = -\sin\theta\,\delta\theta\,\mathbf{I} + \cos\theta\,\delta\theta\,\mathbf{J} = \delta\theta \mathbf{j} \qquad \delta \mathbf{j} = -\cos\theta\,\delta\theta\,\mathbf{I} - \sin\theta\,\delta\theta\,\mathbf{J} = -\delta\theta \mathbf{i} \qquad \textbf{[k]}$$

Equations [k] can also be obtained directly from the rates of change of the unit vectors. Indeed, recalling that the angular velocity is $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, the derivatives of the unit vectors are

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \times \mathbf{i} = \dot{\theta}\mathbf{j} \qquad \frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \times \mathbf{j} = -\dot{\theta}\mathbf{i} \qquad \textbf{[l]}$$

from which the variations can be calculated easily. Introducing Eqs. [k] into Eq. [j], we obtain Eq. [e].

We have thus obtained the variation of $\mathbf{r}_G$ three different ways. It is clear that the number of manipulations is the least when we obtain the variation of $\mathbf{r}_G$ from the velocity expressions.

Example 4.6

Consider the two-link mechanism in Fig. 4.3. A force $\mathbf{F}$ is acting at point P. Find the virtual work expression for each link and demonstrate that Eq. [4.4.14] holds.

Solution

The free-body diagrams of the link are shown in Fig. 4.7. For the first link, the forces that contribute to the virtual work are the reactions at point B and the force of gravity at the mass center G_1. The forces can be expressed in vector form as

$$\mathbf{F}_B = B_x\mathbf{i} + B_y\mathbf{j} \qquad \mathbf{F}_{G1} = -m_1 g\mathbf{j} \qquad \textbf{[a]}$$

The associated displacement vectors are

$$\mathbf{r}_B = L_1\,\mathrm{s}\,\theta_1\mathbf{i} - L_1\,\mathrm{c}\,\theta_1\mathbf{j} \qquad \mathbf{r}_{G_1} = \frac{L_1}{2}\,\mathrm{s}\,\theta_1\mathbf{i} - \frac{L_1}{2}\,\mathrm{c}\,\theta_1\mathbf{j} \qquad \textbf{[b]}$$

so that the virtual displacements become

$$\delta\mathbf{r}_B = L_1\,\mathrm{c}\,\theta_1\,\delta\theta_1\mathbf{i} + L_1\,\mathrm{s}\,\theta_1\,\delta\theta_1\mathbf{j} \qquad \delta\mathbf{r}_{G_1} = \frac{L_1}{2}\,\mathrm{c}\,\theta_1\,\delta\theta_1\mathbf{i} + \frac{L_1}{2}\,\mathrm{s}\,\theta_1\,\delta\theta_1\mathbf{j} \qquad \textbf{[c]}$$

We thus find the virtual work for the first link as

$$\delta W_{\text{link1}} = \mathbf{F}_B \cdot \delta\mathbf{r}_B - m_1 g\mathbf{j} \cdot \delta\mathbf{r}_{G_1} = B_x L_1\,\mathrm{c}\,\theta_1\,\delta\theta + B_y L_1\,\mathrm{s}\,\theta_1\,\delta\theta_1 - \frac{m_1 g L_1}{2}\,\mathrm{s}\,\theta_1\,\delta\theta_1 \qquad \textbf{[d]}$$

For the second link, the virtual work is due to the reactions at point B, gravity acting through the center of mass of the link G_2, and the external force $\mathbf{F}$ at the tip P. The forces at B are equal and opposite of $\mathbf{F}_B$. The other forces can be expressed by

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} \qquad \mathbf{F}_{G_2} = -m_2 g\mathbf{j} \qquad \textbf{[e]}$$

with associated displacements

$$\mathbf{r}_P = (L_1\,\mathrm{s}\,\theta_1 + L_2\,\mathrm{s}\,\theta_2)\mathbf{i} - (L_1\,\mathrm{c}\,\theta_1 + L_2\,\mathrm{c}\,\theta_2)\mathbf{j}$$

$$\mathbf{r}_{G_2} = \left(L_1\,\mathrm{s}\,\theta_1 + \frac{L_2}{2}\,\mathrm{s}\,\theta_2\right)\mathbf{i} - \left(L_1\,\mathrm{c}\,\theta_1 + \frac{L_2}{2}\,\mathrm{c}\,\theta_2\right)\mathbf{j} \qquad \textbf{[f]}$$

whose variations are

$$\delta \mathbf{r}_P = (L_1 \,\mathrm{c}\,\theta_1 \delta\theta_1 + L_2 \,\mathrm{c}\,\theta_2 \delta\theta_2)\mathbf{i} + (L_1 \,\mathrm{s}\,\theta_1 \delta\theta_1 + L_2 \,\mathrm{s}\,\theta_2 \delta\theta_2)\mathbf{j}$$

$$\delta \mathbf{r}_{G_2} = \left(L_1 \,\mathrm{c}\,\theta_1 \delta\theta_1 + \frac{L_2}{2} \,\mathrm{c}\,\theta_2 \delta\theta_2\right)\mathbf{i} + \left(L_1 \,\mathrm{s}\,\theta_1 \delta\theta_1 + \frac{L_2}{2} \,\mathrm{s}\,\theta_2 \delta\theta_2\right)\mathbf{j} \qquad \textbf{[g]}$$

We now find the virtual work for the second link as

$$\delta W_{\text{link2}} = -\mathbf{F}_B \cdot \delta \mathbf{r}_B + \mathbf{F} \cdot \delta \mathbf{r}_P - m_2 g \mathbf{j} \cdot \delta \mathbf{r}_{G_2} \qquad \textbf{[h]}$$

The virtual work of the entire system is found by adding Eqs. [d] and [h], with the result

$$\delta W = \delta W_{\text{link1}} + \delta W_{\text{link2}} = \mathbf{F} \cdot \delta \mathbf{r}_P - m_2 g \mathbf{j} \cdot \delta \mathbf{r}_{G_2} - m_1 g \mathbf{j} \cdot \delta \mathbf{r}_{G_1} \qquad \textbf{[i]}$$

The only terms that contribute to the virtual work are those associated with the external forces. The contribution of the holonomic constraint forces (in this case the reaction at point B) to the virtual work is zero.

Taking the dot products, we write Eq. [i] in terms of the generalized coordinates as

$$\begin{aligned} \delta W &= F_x(L_1 \,\mathrm{c}\,\theta_1 \,\delta\theta_1 + L_2 \,\mathrm{c}\,\theta_2 \,\delta\theta_2) + F_y(L_1 \,\mathrm{s}\,\theta_1 \,\delta\theta_1 + L_2 \,\mathrm{s}\,\theta_2 \,\delta\theta_2) \\ &\quad - m_2 g\left(L_1 \,\mathrm{s}\,\theta_1 \,\delta\theta_1 + \frac{1}{2} L_2 \,\mathrm{s}\,\theta_2 \,\delta\theta_2\right) - \frac{1}{2} m_1 g L_1 \,\mathrm{s}\,\theta_1 \,\delta\theta_1 \\ &= \left(F_x L_1 \,\mathrm{c}\,\theta_1 + F_y L_1 \,\mathrm{s}\,\theta_1 - m_2 g L_1 \,\mathrm{s}\,\theta_1 - \frac{1}{2} m_1 g L_1 \,\mathrm{s}\,\theta_1\right)\delta\theta_1 \\ &\quad + \left(F_x L_2 \,\mathrm{c}\,\theta_2 + F_y L_2 \,\mathrm{s}\,\theta_2 - \frac{1}{2} m_2 g L_2 \,\mathrm{s}\,\theta_2\right)\delta\theta_2 \end{aligned} \qquad \textbf{[j]}$$

Consider next the problem of having not a pinned joint at point B, but a joint that permits sliding motion, such as the collar shown in Fig. 4.14. Such a joint, as we will see in more detail in Chapter 7, is called a *prismatic joint*. The free-body diagram is illustrated in Fig. 4.15. The friction force at the sliding joint must be considered, and the forces that the two bodies exert on each other are split into two parts: a normal force N and a friction force F_f. Introducing the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ along and perpendicular to the link, we express the normal and

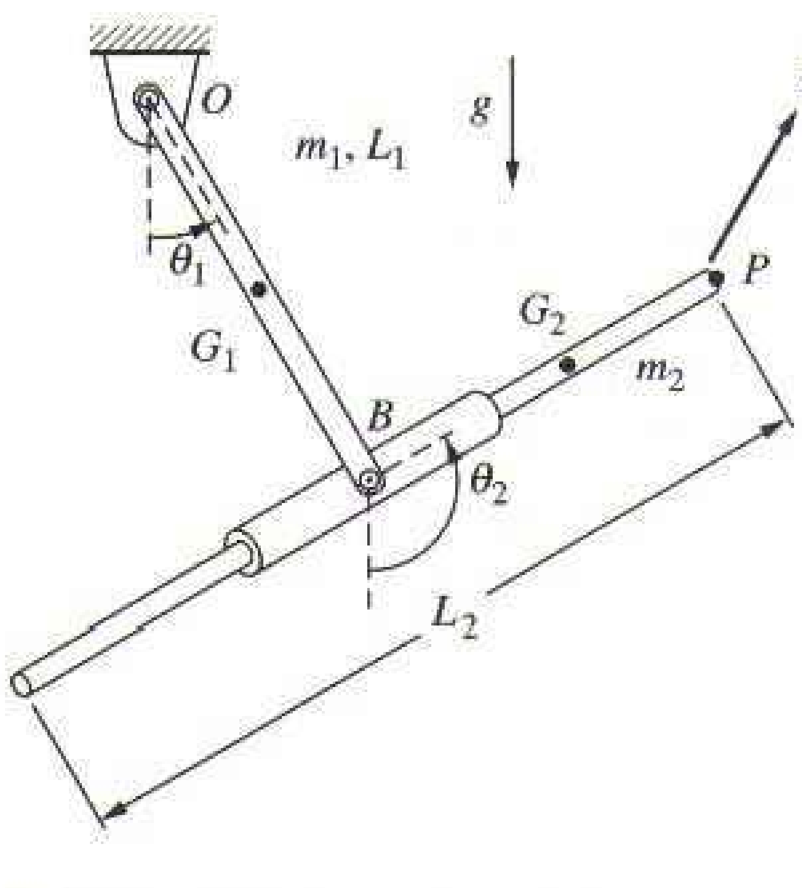

Figure 4.14 Prismatic joint

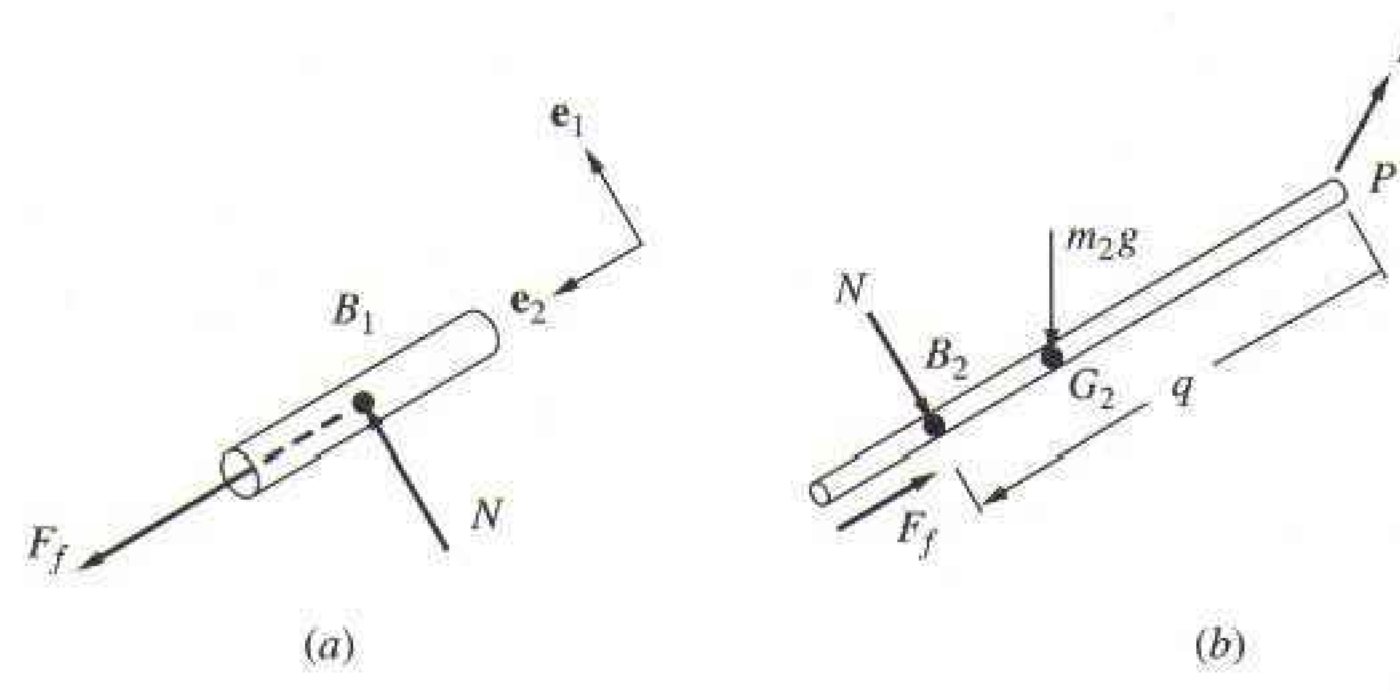

Figure 4.15 Free-body diagrams

friction forces on the two rods as

$$\mathbf{F}_N = N\mathbf{e}_1 \qquad \mathbf{F}_f = F_f\mathbf{e}_2 \qquad \textbf{[k]}$$

Note also that because of sliding, the points B on the first link and on the collar do not have the same velocity. Denoting these points by B_1 and B_2 and introducing a generalized coordinate q to describe the sliding of link 2, the position vectors for B_1 and B_2 become

$$\mathbf{r}_{B_1} = \mathbf{r}_B \qquad \mathbf{r}_{B_2} = \mathbf{r}_B + q\mathbf{e}_2 \qquad \textbf{[l]}$$

The virtual work expression has the form

$$\delta W_{\text{link1}} = (\mathbf{F}_N + \mathbf{F}_f)\cdot\delta\mathbf{r}_{B_1} - m_1 g\mathbf{j}\cdot\delta\mathbf{r}_{G_1}$$

$$\delta W_{\text{link2}} = -(\mathbf{F}_N + \mathbf{F}_f)\cdot\delta\mathbf{r}_{B_2} + \mathbf{F}\cdot\delta\mathbf{r}_P - m_2 g\mathbf{j}\cdot\delta\mathbf{r}_{G_2} \qquad \textbf{[m]}$$

so that the virtual work for the entire system is

$$\delta W = \delta W_{\text{link1}} + \delta W_{\text{link2}} = -F_f\,\delta q + \mathbf{F}\cdot\delta\mathbf{r}_P - m_2 g\mathbf{j}\cdot\delta\mathbf{r}_{G_2} - m_1 g\mathbf{j}\cdot\delta\mathbf{r}_{G_1} \qquad \textbf{[n]}$$

The contribution of the friction force to the virtual work is clear. Note that in order to determine the magnitude of the friction force, we need to have the normal force, which is absent from the above expression. This, basically, is the typical problem encountered when formulating problems involving friction. Also note that the position vectors for the center of mass and for the tip of the second rod change when the sliding joint is introduced to the problem.

4.5 GENERALIZED FORCES

Consider the system of particles in Fig. 4.16. The jth particle exerts a force of $\mathbf{F}_{ij}$ on the ith particle ($i, j = 1, 2, \ldots, N$). The resultant of all forces acting on the ith particle is denoted by $\mathbf{R}_i$ and has the form

$$\mathbf{R}_i = \mathbf{F}_i + \mathbf{F}'_i = \mathbf{F}_i + \sum_{j=1}^{N}\mathbf{F}_{ij} \qquad i = 1, 2, \ldots, N \qquad \textbf{[4.5.1]}$$

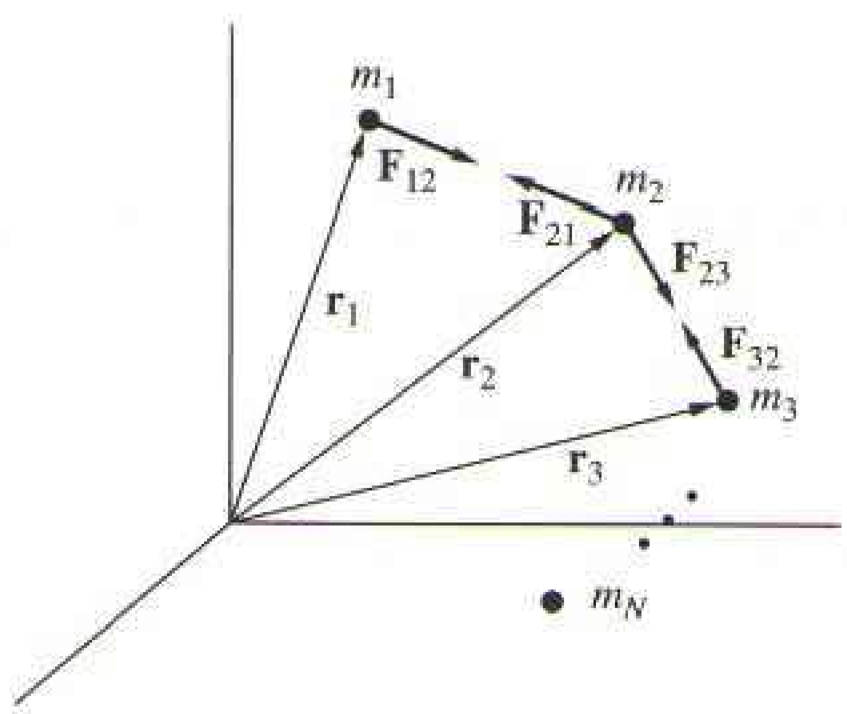

Figure 4.16

where $\mathbf{F}_i$ denotes the sum of all external (impressed, applied) forces exerted on the ith particle and $\mathbf{F}'_i$ is the sum of all internal forces (constraint or reaction forces that one particle exerts on the other).

The virtual work for each particle is defined as

$$\delta W_i = \mathbf{R}_i \cdot \delta \mathbf{r}_i \tag{4.5.2}$$

One obtains the virtual work for the entire system by summing over the individual particles

$$\delta W = \sum_{i=1}^{N} \delta W_i = \sum_{i=1}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i \tag{4.5.3}$$

Substituting Eq. [4.5.1] into Eq. [4.5.3],

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i + \sum_{i=1}^{N} \mathbf{F}'_i \cdot \delta \mathbf{r}_i \tag{4.5.4}$$

We showed in Eq. [4.4.14] that the total work performed by the constraint forces in any virtual displacement is zero. It follows that the second term on the right side of the above equation vanishes because

$$\sum_{i=1}^{N} \mathbf{F}'_i \cdot \delta \mathbf{r}_i = 0 \tag{4.5.5}$$

and the expression for the virtual work becomes

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i \tag{4.5.6}$$

It is of interest to examine the virtual work in terms of generalized coordinates. We express the displacement of each particle in terms of a set of n generalized coordinates q_k $(k = 1, 2, \ldots, n)$ as $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t)$, $(i = 1, 2, \ldots, N)$. The variation of $\mathbf{r}_i$ is

$$\delta \mathbf{r}_i = \sum_{i=1}^{N} \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \tag{4.5.7}$$

Substitution of Eq. [4.5.7] into the expression for virtual work yields

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k \tag{4.5.8}$$

We define the term inside the fences in the above equation as *generalized forces* and write

$$Q_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \qquad k = 1, 2, \ldots, n \tag{4.5.9}$$

where Q_k is the *generalized force associated with the kth generalized coordinate.* We can then express the virtual work as

$$\delta W = \sum_{k=1}^{n} Q_k \, \delta q_k \qquad \text{[4.5.10]}$$

The relation between a generalized coordinate and a generalized force is analogous to the relation between a physical coordinate and the force applied in the direction of that coordinate. Also, the dimensional relation between generalized coordinates and generalized forces is worth noting. The product of Q_k and δq_k has the same units as the variation of energy. For example, if the generalized coordinate describes a displacement, the generalized force has the units of force. If the generalized coordinate describes a rotation, the generalized force becomes a moment.

Recalling from the previous section that the variations are often calculated with more ease by velocity relations, we make use of Eq. [4.4.6]

$$\frac{\partial \mathbf{r}}{\partial q_k} = \frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_k} \qquad k = 1, 2, \ldots, n \qquad \text{[4.5.11]}$$

to express the generalized forces as

$$Q_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \qquad \text{[4.5.12]}$$

Another way of calculating generalized forces is based on the nature of the applied forces. For a conservative system, because dW is a perfect differential, the virtual work can be written as the variation of the negative of the potential energy, or

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = -\delta V \qquad \text{[4.5.13]}$$

in which V is the *potential function,* or the *potential energy.* The variation of the potential energy in terms of physical coordinates is

$$\delta V = \sum_{i=1}^{N} \left(\frac{\partial V}{\partial x_i} \delta x_i + \frac{\partial V}{\partial y_i} \delta y_i + \frac{\partial V}{\partial z_i} \delta z_i \right) \qquad \text{[4.5.14]}$$

When there are no constraints acting on the system, x_i, y_i, and z_i ($i = 1, 2, \ldots, N$) are independent. It follows that δx_i, δy_i and δz_i are arbitrary, and using Eqs. [4.5.13] and [4.5.14], we obtain

$$\frac{\partial V}{\partial x_i} = -F_{x_i} \qquad \frac{\partial V}{\partial y_i} = -F_{y_i} \qquad \frac{\partial V}{\partial z_i} = -F_{z_i} \qquad \text{[4.5.15]}$$

In terms of independent generalized coordinates, and when all the applied forces are conservative, the virtual work expression can be written as

$$\delta V = -\delta W = \sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \delta q_k \qquad \text{[4.5.16]}$$

Comparing Eqs. [4.5.16] and [4.5.10], and considering the independence of the variations of the generalized coordinates, we conclude that the generalized forces are related to the potential energy by

$$Q_k = -\frac{\partial V}{\partial q_k} \tag{4.5.17}$$

In the presence of both conservative and nonconservative forces, the virtual work and generalized forces can be written as

$$\delta W = -\delta V + \delta W_{nc} \tag{4.5.18}$$

$$Q_k = Q_{kc} + Q_{knc} = -\frac{\partial V}{\partial q_k} + \sum_{i=1}^{N} \mathbf{F}_{inc} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \tag{4.5.19}$$

where the notation is obvious. When they are constant, nonconservative forces can be treated as conservative.

In summary, one can use a number of ways to calculate generalized forces:

1. Write Eq. [4.5.6] and after the virtual work is calculated collect coefficients of δq_k $(k = 1, 2, \ldots, n)$.
2. Calculate $\partial \mathbf{r}_i/\partial q_k$ (or $\partial \dot{\mathbf{r}}_i/\partial \dot{q}_k$) and use Eq. [4.5.12].
3. Take advantage of the potential energy and use Eq. [4.5.17] for the conservative forces.

The reader is encouraged to use and compare all three approaches.

Example 4.7

Find the generalized forces for the mechanism in Fig. 4.3 (Example 4.6).

Solution

The generalized coordinates are θ_1 and θ_2. We will calculate the generalized forces in a number of ways. First, we take the expression for virtual work from Eq. [j] in Example 4.6, thus

$$\delta W = \left(F_x L_1 \,\mathrm{c}\,\theta_1 + F_y L_1 \,\mathrm{s}\,\theta_1 - m_2 g L_1 \,\mathrm{s}\,\theta_1 - m_1 g \frac{L_1}{2} \,\mathrm{s}\,\theta_1\right)\delta\theta_1$$
$$+ \left(F_x L_2 \,\mathrm{c}\,\theta_2 + F_y L_2 \,\mathrm{s}\,\theta_2 - m_2 g \frac{L_2}{2} \,\mathrm{s}\,\theta_2\right)\delta\theta_2 \tag{a}$$

so that we can identify the generalized forces as

$$Q_1 = F_x L_1 \,\mathrm{c}\,\theta_1 + F_y L_1 \,\mathrm{s}\,\theta_1 - m_2 g L_1 \,\mathrm{s}\,\theta_1 - \frac{1}{2} m_1 g L_1 \,\mathrm{s}\,\theta_1$$
$$Q_2 = F_x L_2 \,\mathrm{c}\,\theta_2 + F_y L_2 \,\mathrm{s}\,\theta_2 - \frac{1}{2} m_2 g L_2 \,\mathrm{s}\,\theta_2 \tag{b}$$

Next, consider each force and $\partial \mathbf{r}_i/\partial q_k$. For the first link we have one external force, gravity, and

$$\mathbf{F}_{G_1} = -m_1 g \mathbf{j} \qquad r_{G_1} = \frac{L_1}{2} \,\mathrm{s}\,\theta_1 \mathbf{i} - \frac{L_1}{2} \,\mathrm{c}\,\theta_1 \mathbf{j} \tag{c}$$

so that

$$\frac{\partial \mathbf{r}_{G_1}}{\partial \theta_1} = \frac{L_1}{2}\,\mathrm{c}\,\theta_1\mathbf{i} + \frac{L_1}{2}\,\mathrm{s}\,\theta_1\mathbf{j} \qquad \frac{\partial \mathbf{r}_{G_1}}{\partial \theta_2} = \mathbf{0} \tag{d}$$

There are two external forces acting on the second link, written

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} \qquad \mathbf{r}_P = (L_1\,\mathrm{s}\,\theta_1 + L_2\,\mathrm{s}\,\theta_2)\mathbf{i} - (L_1\,\mathrm{c}\,\theta_1 + L_2\,\mathrm{c}\,\theta_2)\mathbf{j}$$

$$\mathbf{F}_{G_2} = -m_2 g\mathbf{j} \qquad \mathbf{r}_{G_2} = \left(L_1\,\mathrm{s}\,\theta_1 + \frac{L_2}{2}\,\mathrm{s}\,\theta_2\right)\mathbf{i} - \left(L_1\,\mathrm{c}\,\theta_1 + \frac{L_2}{2}\,\mathrm{c}\,\theta_2\right)\mathbf{j} \tag{e}$$

so that

$$\frac{\partial \mathbf{r}_P}{\partial \theta_1} = L_1\,\mathrm{c}\,\theta_1\mathbf{i} + L_1\,\mathrm{s}\,\theta_1\mathbf{j} \qquad \frac{\partial \mathbf{r}_P}{\partial \theta_2} = L_2\,\mathrm{c}\,\theta_2\mathbf{i} + L_2\,\mathrm{s}\,\theta_2\mathbf{j}$$

$$\frac{\partial \mathbf{r}_{G_2}}{\partial \theta_1} = L_1\,\mathrm{c}\,\theta_1\mathbf{i} + L_1\,\mathrm{s}\,\theta_1\mathbf{j} \qquad \frac{\partial \mathbf{r}_{G_2}}{\partial \theta_2} = \frac{1}{2}L_2\,\mathrm{c}\,\theta_2\mathbf{i} + \frac{1}{2}L_2\,\mathrm{s}\,\theta_2\mathbf{j} \tag{f}$$

Applying Eq. [4.5.12] we obtain

$$\begin{aligned} Q_1 &= \mathbf{F}_{G_1}\cdot\frac{\partial \mathbf{r}_{G_1}}{\partial \theta_1} + \mathbf{F}\cdot\frac{\partial \mathbf{r}_P}{\partial \theta_1} + \mathbf{F}_{G_2}\cdot\frac{\partial \mathbf{r}_{G_2}}{\partial \theta_1} \\ &= -\frac{1}{2}m_1 g L_1\,\mathrm{s}\,\theta_1 - m_2 g L_1\,\mathrm{s}\,\theta_1 + F_x L_1\,\mathrm{c}\,\theta_1 + F_y L_1\,\mathrm{s}\,\theta_1 \\ Q_2 &= \mathbf{F}_{G_1}\cdot\frac{\partial \mathbf{r}_{G_1}}{\partial \theta_2} + \mathbf{F}\cdot\frac{\partial \mathbf{r}_P}{\partial \theta_2} + \mathbf{F}_{G_2}\cdot\frac{\partial \mathbf{r}_{G_2}}{\partial \theta_2} \\ &= -\frac{1}{2}m_2 g L_2\,\mathrm{s}\,\theta_2 + F_x L_2\,\mathrm{c}\,\theta_2 + F_y L_2\,\mathrm{s}\,\theta_2 \end{aligned} \tag{g}$$

which are the same as Eq. [b].

Finally, we make use of the potential energy to calculate the portion of the generalized forces associated with the gravitational forces. Taking point O as the datum, we write the potential energy as

$$V = -m_1 g\frac{L_1}{2}\,\mathrm{c}\,\theta_1 - m_2 g L_1\,\mathrm{c}\,\theta_1 - m_2 g\frac{L_2}{2}\,\mathrm{c}\,\theta_2 \tag{h}$$

hence, the generalized forces due to the conservative forces become

$$\begin{aligned} Q_{1c} &= -\frac{\partial V}{\partial \theta_1} = -\frac{1}{2}m_1 g L_1\,\mathrm{s}\,\theta_1 - m_2 g L_1\,\mathrm{s}\,\theta_1 \\ Q_{2c} &= -\frac{\partial V}{\partial \theta_2} = -\frac{1}{2}m_2 g L_2\,\mathrm{s}\,\theta_2 \end{aligned} \tag{i}$$

It is easy to see that the use of potential energy simplifies the calculation of the generalized forces.

4.6 PRINCIPLE OF VIRTUAL WORK FOR STATIC EQUILIBRIUM

Let us now consider *static equilibrium*. For a dynamical system, static equilibrium is described as the state where all components of the system are at rest, with zero velocity and zero acceleration. To find the equilibrium position, one can write the equilibrium equations using Newton's second law and solve these equations. The disadvantage of doing so is that if the motions of any two components are related to each other with a constraint relation, then the associated constraint forces must be calculated in the process. This may become tedious for systems with several interconnected components. Encouraged by the results of the previous section, we seek a different solution to the equilibrium problem that does not require one to solve for the constraint equations.

At equilibrium, the resultant force on each component of a system must be zero. Hence, we have $\mathbf{R}_i = \mathbf{0}\, (i = 1, 2, \ldots, N)$. It follows from Eq. [4.5.3] that since every resultant $\mathbf{R}_i = \mathbf{0}$, the virtual work must vanish as well and we must have $\delta W = 0$. Introducing this into Eq. [4.5.6] gives

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \qquad \textbf{[4.6.1]}$$

The above equation, first formulated by Johann Bernoulli, is known as the *principle of virtual work for static equilibrium.* It basically states that, at static equilibrium, the work performed by the external, impressed forces through virtual displacements compatible with the system constraints is zero. It can easily be extended to rigid bodies if we consider $\mathbf{r}_i$ to be the displacement of the point on the body to which the force $\mathbf{F}_i$ is applied.

Let us consider the principle of virtual work in terms of generalized forces. It follows from Eq. [4.5.10] that at equilibrium

$$\delta W = \sum_{k=1}^{n} Q_k \, \delta q_k = 0 \qquad \textbf{[4.6.2]}$$

When the system is represented in terms of independent generalized coordinates, because the generalized coordinates are independent of each other, their variations δq_k also are independent. Therefore, for Eq. [4.6.2] to hold, each of the coefficients of δq_k, that is, Q_k, must vanish individually. We write

$$Q_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = 0 \qquad k = 1, 2, \ldots, n \qquad \textbf{[4.6.3]}$$

In the presence of conservative forces we can take advantage of the potential energy and write

$$-\frac{\partial V}{\partial q_k} + Q_{knc} = 0 \qquad \textbf{[4.6.4]}$$

The above results can also be interpreted as follows: Because independent generalized coordinates represent the independent motion of each degree of freedom, their corresponding generalized forces must vanish at equilibrium.

As in the previous section, one can follow two approaches when solving static equilibrium problems using the principle of virtual work:

1. One can work with physical coordinates and use Eq. [4.6.1].
2. One can select a set of generalized coordinates, calculate the associated generalized forces, and use Eq. [4.6.3] or [4.6.4].

In the second approach, Eq. [4.6.4] is usually recommended over Eq. [4.6.3] in the presence of conservative forces, as it makes use of the potential energy. On the other hand, computation of $\partial \mathbf{r}_i/\partial q_k$ or $\partial \dot{\mathbf{r}}_i/\partial \dot{q}_k$ ($i = 1, 2, \ldots, N$; $k = 1, 2, \ldots, n$) in Eq. [4.6.3] can be done in a systematic fashion and tabulated, thereby mechanizing the derivation of the equilibrium equations.

Next, consider the principle of virtual work in terms of constrained generalized coordinates. To this end, write the constraint equations in Pfaffian form as

$$\sum_{k=1}^{n} a_{jk}\, dq_k + a_{j0}\, dt = 0 \qquad j = 1, 2, \ldots, m \tag{4.6.5}$$

Now write the variation of the generalized coordinates as

$$\sum_{k=1}^{n} a_{jk}\, \delta q_k = 0 \tag{4.6.6}$$

We add this relation to the principle of virtual work via the Lagrange multipliers λ_j ($j = 1, 2, \ldots, m$), resulting in the expression for the augmented virtual work as

$$\delta \hat{W} = \delta W - \sum_{j=1}^{m} \lambda_j \left(\sum_{k=1}^{n} a_{jk}\, \delta q_k \right) = \sum_{k=1}^{n} Q_k\, \delta q_k - \sum_{j=1}^{m} \lambda_j \left(\sum_{k=1}^{n} a_{jk}\, \delta q_k \right) = 0 \tag{4.6.7}$$

Rearranging this equation as

$$\delta \hat{W} = \sum_{k=1}^{n} \left(Q_k - \sum_{j=1}^{m} \lambda_j a_{jk} \right) \delta q_k = 0 \tag{4.6.8}$$

and by selecting the Lagrange multipliers such that the coefficients of δq_k vanish individually, we write the equilibrium equations as

$$Q_k = \sum_{j=1}^{m} \lambda_j a_{jk} \qquad k = 1, 2, \ldots, n \tag{4.6.9}$$

In the presence of conservative forces, we introduce Eq. [4.6.4] to this equation, which leads to

$$Q_{knc} = \frac{\partial V}{\partial q_k} + \sum_{j=1}^{m} \lambda_j a_{jk} \tag{4.6.10}$$

Example 4.8

Find the equilibrium position of the two links in Fig. 4.17. The springs are unstretched when both rods are horizontal. Both springs deflect only vertically.

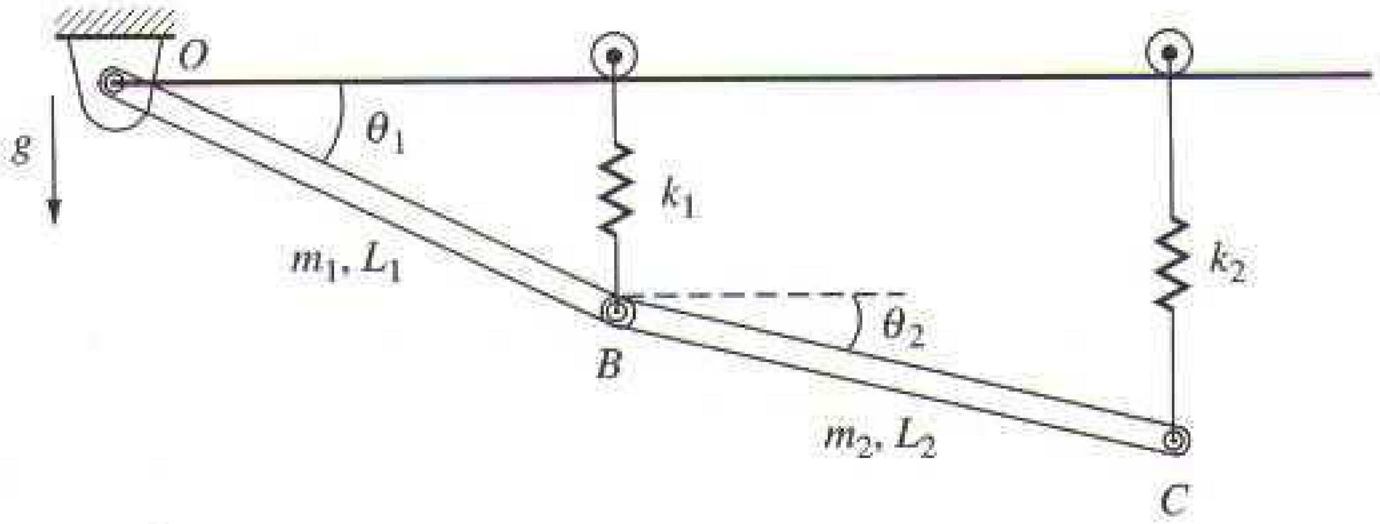

Figure 4.17

Solution

Because this problem involves two interconnected bodies and it is conservative, it is preferable to use potential energy to find the equilibrium position. Noting that the spring deflections are $L_1 \sin\theta_1$ and $L_1 \sin\theta_1 + L_2 \sin\theta_2$, and taking as the datum position the horizontal position of both links, the potential energy is

$$V = -m_1 g \frac{L_1}{2} \mathrm{s}\,\theta_1 + \frac{1}{2} k_1 (L_1 \mathrm{s}\,\theta_1)^2 - m_2 g \left(L_1 \mathrm{s}\,\theta_1 + \frac{L_2}{2} \mathrm{s}\,\theta_2 \right) + \frac{1}{2} k_2 (L_1 \mathrm{s}\,\theta_1 + L_2 \mathrm{s}\,\theta_2)^2 \qquad \textbf{[a]}$$

The equilibrium positions are found from

$$\frac{\partial V}{\partial \theta_1} = 0 \qquad \frac{\partial V}{\partial \theta_2} = 0 \qquad \textbf{[b]}$$

and, taking the partial derivatives of V, we obtain

$$\frac{\partial V}{\partial \theta_1} = -\frac{1}{2} m_1 g L_1 \mathrm{c}\,\theta_1 + k_1 L_1^2 \mathrm{s}\,\theta_1 \mathrm{c}\,\theta_1 - m_2 g L_1 \mathrm{c}\,\theta_1 + k_2 (L_1 \mathrm{s}\,\theta_1 + L_2 \mathrm{s}\,\theta_2) L_1 \mathrm{c}\,\theta_1 \qquad \textbf{[c]}$$

$$\frac{\partial V}{\partial \theta_2} = -\frac{1}{2} m_2 g L_2 \mathrm{c}\,\theta_2 + k_2 (L_1 \mathrm{s}\,\theta_1 + L_2 \mathrm{s}\,\theta_2) L_2 \mathrm{c}\,\theta_2$$

We introduce Eqs. [c] into Eqs. [b]. Because $\cos\theta_1$ and $\cos\theta_2$ are common to the first and second of Eqs. [c], respectively, we eliminate them from Eqs. [c] and obtain

$$(k_1 + k_2) L_1^2 \mathrm{s}\,\theta_1 + k_2 L_1 L_2 \mathrm{s}\,\theta_2 = \frac{1}{2} m_1 g L_1 + m_2 g L_1 \qquad \textbf{[d]}$$

$$k_2 L_1 L_2 \mathrm{s}\,\theta_1 + k_2 L_2^2 \mathrm{s}\,\theta_2 = \frac{1}{2} m_2 g L_2$$

Note that by eliminating $\cos\theta_1$ and $\cos\theta_2$ from the formulation, we are concluding that $\cos\theta_1 = 0$ and $\cos\theta_2 = 0$ represent equilibrium positions themselves. This basically is the vertical position of the links. At equilibrium either both links can be vertical, or one can. If link 1 is vertical, then the equilibrium position for link 2 is found by solving the second of Eqs. [d], and vice versa. To find the equilibrium positions where neither link is vertical, we solve Eqs. [d] simultaneously. To this end, we express Eqs. [d] in matrix form by

$$\begin{bmatrix} (k_1 + k_2) L_1^2 & k_2 L_1 L_2 \\ k_2 L_1 L_2 & k_2 L_2^2 \end{bmatrix} \begin{bmatrix} \sin\theta_1 \\ \sin\theta_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} m_1 g L_1 + m_2 g L_1 \\ \frac{1}{2} m_2 g L_2 \end{bmatrix} \qquad \textbf{[e]}$$

which can be written as $[K]\{q\} = \{Q\}$, and whose solution is $\{q\} = [K]^{-1}\{Q\}$. The solution can be shown to be

$$\sin\theta_1 = \frac{g}{2k_1L_1}(m_1 + m_2) \qquad \textbf{[f]}$$

$$\sin\theta_2 = -\frac{g}{2k_1L_2}(m_1 + m_2) + \frac{g}{2k_2L_2}m_2$$

An interesting case arises when k_1 is set to zero, or when there is no spring attached to the middle link. In this case, $\det[K] = 0$, which implies that one cannot solve for the equilibrium position by inverting Eq. [e]. The double link can assume an infinite number of equilibrium positions.

4.7 D'ALEMBERT'S PRINCIPLE

D'Alembert's principle extends the principle of virtual work from the static to the dynamic case. Consider the system of N particles discussed in the previous sections. If the system is not at rest, we can write Newton's second law for the ith particle as

$$\mathbf{R}_i = m_i\mathbf{a}_i = \frac{d}{dt}\mathbf{p}_i \qquad i = 1, 2, \ldots, N \qquad \textbf{[4.7.1]}$$

where $\mathbf{p}_i = m\mathbf{v}_i$ is the linear momentum of the ith particle and $\mathbf{R}_i$ is the resultant of all forces acting on the ith particle. As in the static case, we split the resultant $\mathbf{R}_i$ into the sum of the externally applied and constraint forces as

$$\mathbf{R}_i = \mathbf{F}_i + \mathbf{F}_i' \qquad \textbf{[4.7.2]}$$

Introducing Eq. [4.7.2] into Eq. [4.7.1], we obtain

$$\mathbf{F}_i + \mathbf{F}_i' - \dot{\mathbf{p}}_i = \mathbf{0} \qquad \textbf{[4.7.3]}$$

This equation is known as the *dynamic equilibrium* relation, where the negative of the rate of change of linear momentum, $-\dot{\mathbf{p}}_i = -m_i\mathbf{a}_i$, is treated as a force, referred to as the *inertia force*, that provides equilibrium. We can now treat the dynamic system as if it is a static system and invoke the principle of virtual work. Equation [4.7.3] is sometimes referred to as D'Alembert's principle. We proceed with the dot product of Eq. [4.7.3] and the variation in the displacement, and write

$$(\mathbf{F}_i + \mathbf{F}_i' - m_i\mathbf{a}_i) \cdot \delta\mathbf{r}_i = 0 \qquad \textbf{[4.7.4]}$$

Summing over all the particles gives

$$\sum_{i=1}^{N}(\mathbf{F}_i + \mathbf{F}_i' - m_i\mathbf{a}_i) \cdot \delta\mathbf{r}_i = 0 \qquad \textbf{[4.7.5]}$$

Recalling from Section 4.4 that work done by the constraint forces over virtual displacements is zero, or

$$\sum_{i=1}^{N} \mathbf{F}_i' \cdot \delta \mathbf{r}_i = 0 \qquad [4.7.6]$$

and subtracting Eq. [4.7.6] from Eq. [4.7.5], we arrive at

$$\sum_{i=1}^{N} (\mathbf{F}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{r}_i = 0 \qquad [4.7.7]$$

This we call the *generalized principle of D'Alembert,* or *D'Alembert's principle.* We observe immediately that the principle of virtual work, given in Eq. [4.6.1], becomes a special case of D'Alembert's principle.

D'Alembert's principle is a fundamental principle that provides a complete formulation of all of the problems of mechanics. Hamilton's principle and Lagrange's equations are all derived from D'Alembert's principle, as will be shown in the next sections. The advantage of using D'Alembert's principle over a Newtonian approach is that constraint forces and interacting forces between particles are eliminated from the formulation. This advantage becomes more pronounced for systems with several degrees of freedom.

We next extend D'Alembert's principle to rigid bodies. We consider here plane motion only (the general three-dimensional case will be derived in Chapter 8). We treat a rigid body as a collection of particles, so that in Eq. [4.7.7], N approaches infinity. Define the angular velocity of the rigid body as $\omega = \dot{\theta}$. Also, we express the position, velocity, and acceleration in terms of the center of mass motion as

$$\mathbf{r}_i = \mathbf{r}_G + \boldsymbol{\rho}_i \qquad \mathbf{v}_i = \mathbf{v}_G + \boldsymbol{\omega} \times \boldsymbol{\rho}_i$$
$$\mathbf{a}_i = \mathbf{a}_G + \boldsymbol{\alpha} \times \boldsymbol{\rho}_i - \omega^2 \boldsymbol{\rho}_i \qquad i = 1, 2, \ldots, N \qquad [4.7.8]$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, $\boldsymbol{\alpha} = \ddot{\theta}\mathbf{k}$, so that the variation of $\mathbf{r}_i$ can be written as

$$\delta \mathbf{r}_i = \delta \mathbf{r}_G + \delta\theta \mathbf{k} \times \boldsymbol{\rho}_i \qquad [4.7.9]$$

and we recognize that $\boldsymbol{\delta\theta} = \delta\theta\, \mathbf{k}$. Introducing Eqs. [4.7.8] and [4.7.9] into D'Alembert's principle, we obtain

$$\sum_{i=1}^{N} (\mathbf{F}_i - m_i \mathbf{a}_G - m_i \boldsymbol{\alpha} \times \boldsymbol{\rho}_i + m_i \omega^2 \boldsymbol{\rho}_i) \cdot (\delta \mathbf{r}_G + \delta\theta\, \mathbf{k} \times \boldsymbol{\rho}_i) = 0 \qquad [4.7.10]$$

Now, consider that the number of particles approaches infinity. The summation is replaced by integration, and m_i, $\boldsymbol{\rho}_i$, and $\mathbf{F}_i$ are replaced by dm, $\boldsymbol{\rho}$, and $d\mathbf{F}$, respectively. Evaluating the individual terms and using the definitions of center of mass and mass moment of inertia, we obtain

$$\int d\mathbf{F} \cdot \delta \mathbf{r}_G = \mathbf{F} \cdot \delta \mathbf{r}_G \qquad \int d\mathbf{F} \cdot (\delta\theta \mathbf{k} \times \boldsymbol{\rho}) = M_G\, \delta\theta \qquad \int dm \mathbf{a}_G \cdot \delta \mathbf{r}_G = m \mathbf{a}_G \cdot \delta \mathbf{r}_G$$
$$\int dm (\ddot{\theta}\, \mathbf{k} \times \boldsymbol{\rho}) \cdot (\delta\theta\, \mathbf{k} \times \boldsymbol{\rho}) = \ddot{\theta}\, \delta\theta \int \rho^2\, dm = I_G \ddot{\theta}\, \delta\theta \qquad [4.7.11]$$

where m is the total mass, $\mathbf{F}$ is the resultant of all forces, I_G is the centroidal mass moment of inertia, and M_G is the sum of moments about the center of mass. All other remaining terms in Eq. [4.7.10] are zero. It follows that D'Alembert's principle for a rigid body in plane motion is

$$(\mathbf{F} - m\mathbf{a}_G) \cdot \delta\mathbf{r}_G + (M_G - I_G\ddot{\theta})\,\delta\theta = 0 \qquad \textbf{[4.7.12]}$$

For a system of N rigid bodies in plane motion, D'Alembert's principle becomes

$$\sum_{i=1}^{N}[(\mathbf{F}_i - m_i\mathbf{a}_{G_i}) \cdot \delta\mathbf{r}_{G_i} + (M_{G_i} - I_{G_i}\ddot{\theta}_i)\,\delta\theta_i] = 0 \qquad \textbf{[4.7.13]}$$

where the subscript i now denotes the ith rigid body.

Up until the second half of the 20th century, the property of D'Alembert's principle being a vector relationship was usually viewed as a disadvantage, and D'Alembert's principle was primarily considered as a tool to obtain Hamilton's principle and Lagrange's equations. Equations [4.7.7] or [4.7.13] were rarely used in the form given here. The need to deal with complex multibody problems and the availability of digital computers has led scientists and engineers to take another look at D'Alembert's principle as a primary method of solution. For example, if we introduce Eq. [4.4.1b] into Eq. [4.7.7], we obtain

$$\sum_{i=1}^{N}(\mathbf{F}_i - m_i\mathbf{a}_i) \cdot \left(\sum_{k=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_k}\,\delta q_k\right) = \sum_{k=1}^{n}\left[\sum_{i=1}^{N}(\mathbf{F}_i - m_i\mathbf{a}_i) \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right]\delta q_k = 0 \qquad \textbf{[4.7.14]}$$

When we have a set of independent generalized coordinates, the coefficients of δq_k must vanish independently, with the result

$$\sum_{i=1}^{N}(\mathbf{F}_i - m_i\mathbf{a}_i) \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = 0 \qquad k = 1, 2, \ldots, n \qquad \textbf{[4.7.15]}$$

Extending this to the case of N rigid bodies in plane motion, we obtain

$$\sum_{i=1}^{N}\left[(\mathbf{F}_i - m_i\mathbf{a}_{G_i}) \cdot \frac{\partial \mathbf{r}_{G_i}}{\partial q_k} + (M_{G_i} - I_{G_i}\ddot{\theta}_i)\frac{\partial \theta_i}{\partial q_k}\right] = 0 \qquad \textbf{[4.7.16]}$$

Equations [4.7.15] and [4.7.16] represent direct use of D'Alembert's principle to derive equations of motion.

Example 4.9

Consider a bead of mass m free to slide on a ring (hoop) of radius R, as shown in Fig. 4.18. The ring is rotating with the constant angular velocity Ω. Find the equation of motion using D'Alembert's principle.

Solution

Because we are dealing with a single particle, we drop the subscript in Eq. [4.7.7] and write it as

$$(\mathbf{F} - m\mathbf{a}) \cdot \delta\mathbf{r} = 0 \qquad \textbf{[a]}$$

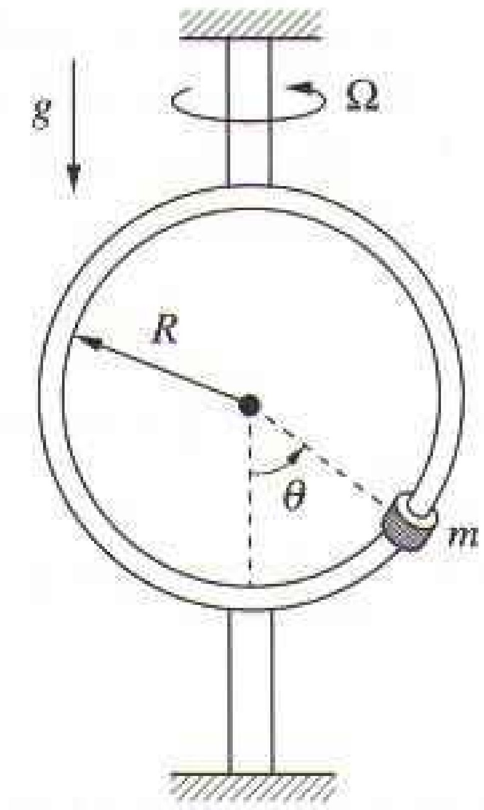

Figure 4.18 Bead on a rotating ring

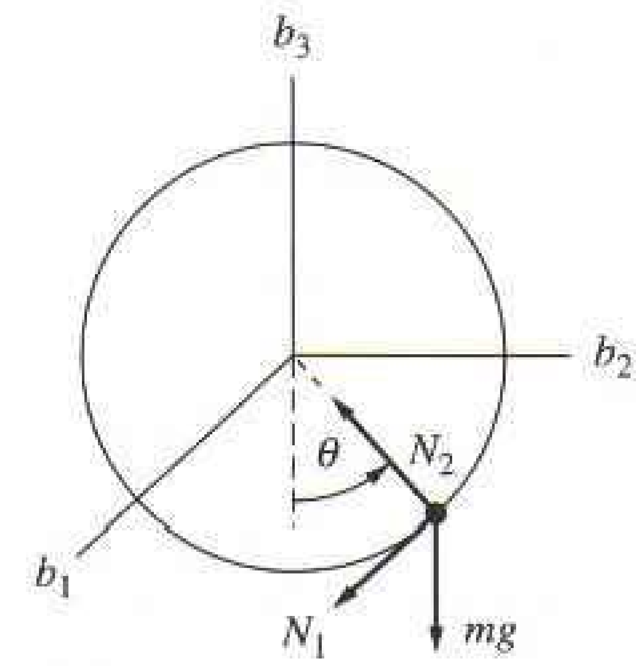

Figure 4.19 Free-body diagram

The free-body diagram is given in Fig. 4.19. The $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ axes are attached to the hoop. The generalized coordinate is selected as θ. We first derive an expression for the acceleration. The moving frame is attached to the ring. The position vector is

$$\mathbf{r} = R\sin\theta\,\mathbf{b}_2 - R\cos\theta\,\mathbf{b}_3 \qquad \textbf{[b]}$$

so its variation is

$$\delta\mathbf{r} = R\cos\theta\,\delta\theta\,\mathbf{b}_2 + R\sin\theta\,\delta\theta\,\mathbf{b}_3 \qquad \textbf{[c]}$$

Because the motion of the relative frame, that is, of the hoop, is treated as a known quantity, its variation is zero. Hence, it is possible to calculate the variation of $\mathbf{r}$ in the relative frame. To see this better, write the velocity of the bead as

$$\mathbf{v} = \mathbf{v}_{\text{rel}} + \boldsymbol{\omega}\times\mathbf{r} = R\dot{\theta}\cos\theta\mathbf{b}_2 + R\dot{\theta}\sin\theta\mathbf{b}_3 + \Omega\mathbf{b}_3\times(R\sin\theta\mathbf{b}_2 - R\cos\theta\mathbf{b}_3) \qquad \textbf{[d]}$$

$$= R\dot{\theta}\cos\theta\mathbf{b}_2 + R\dot{\theta}\sin\theta\mathbf{b}_3 - R\Omega\sin\theta\mathbf{b}_1$$

Since Ω is a constant and it is not the derivative of a motion variable, it cannot be expressed in terms of a variation. Consequently, the third term on the right side of Eq. [d] does not contribute to the virtual displacement.

Because the angular velocity is constant, the expression for the acceleration has the form

$$\mathbf{a} = \mathbf{a}_{\text{rel}} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r} + 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}} = R\ddot{\theta}\cos\theta\mathbf{b}_2 + R\ddot{\theta}\sin\theta\mathbf{b}_3 - R\dot{\theta}^2\sin\theta\mathbf{b}_2 + R\dot{\theta}^2\cos\theta\mathbf{b}_3$$

$$+ \Omega\mathbf{b}_3\times\Omega\mathbf{b}_3\times(R\sin\theta\mathbf{b}_2 - R\cos\theta\mathbf{b}_3) + 2\Omega\mathbf{b}_3\times(R\dot{\theta}\cos\theta\mathbf{b}_2 + R\dot{\theta}\sin\theta\mathbf{b}_3)$$

$$\mathbf{a} = -2R\dot{\theta}\Omega\cos\theta\mathbf{b}_1 + (-R\sin\theta(\dot{\theta}^2+\Omega^2) + R\ddot{\theta}\cos\theta)\mathbf{b}_2 + (R\dot{\theta}^2\cos\theta + R\ddot{\theta}\sin\theta)\mathbf{b}_3 \qquad \textbf{[e]}$$

The only force acting on the system which is not a constraint force is gravity, and it has the form $\mathbf{F} = -mg\mathbf{b}_3$.

Substituting Eqs. [c] and [e] into the generalized principle of D'Alembert yields

$$(\mathbf{F} - m\mathbf{a})\cdot\delta\mathbf{r} = [-mg\mathbf{b}_3 + m(R\sin\theta(\dot{\theta}^2+\Omega^2) - R\ddot{\theta}\cos\theta)\mathbf{b}_2$$

$$- m(R\dot{\theta}^2\cos\theta + R\ddot{\theta}\sin\theta)\mathbf{b}_3 - 2mR\dot{\theta}\Omega\cos\theta\,\mathbf{b}_1]$$

$$\cdot(R\cos\theta\,\delta\theta\,\mathbf{b}_2 + R\sin\theta\,\delta\theta\,\mathbf{b}_3) = 0 \qquad \textbf{[f]}$$

After evaluating the dot product and setting the coefficient of $\delta\theta$ equal to zero, we obtain the equation of motion as

$$\ddot{\theta} + \sin\theta\left(\frac{g}{R} - \Omega^2\cos\theta\right) = 0 \qquad \textbf{[g]}$$

Let us compare the procedure we used in this example with a Newtonian approach. From the free-body diagram, there are two normal (reaction) forces, N_1 and N_2. After applying Newton's second law, we get three equations and we need to eliminate the reactions. It is obvious that using D'Alembert's principle is simpler. The difference becomes more pronounced where there are several degrees of freedom.

4.8 HAMILTON'S PRINCIPLES

From D'Alembert's principle we develop the scalar variational principles that provide a complete formulation of the problems of mechanics. These principles were stated for the most general case of motion by Sir William Rowan Hamilton.

Consider a system of N particles and D'Alembert's principle

$$\sum_{i=1}^{N}(m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i)\cdot\delta\mathbf{r}_i = 0 \qquad \textbf{[4.8.1]}$$

We denote by $\delta W = \sum \mathbf{F}_i\cdot\delta\mathbf{r}_i$ the virtual work of all the impressed forces. To manipulate the first term in the above equation, consider the expression

$$\frac{d}{dt}(\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i) = \ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + \dot{\mathbf{r}}_i\cdot\delta\dot{\mathbf{r}}_i \qquad i = 1, 2, \ldots, N \qquad \textbf{[4.8.2]}$$

The second term on the right in Eq. [4.8.2] can be recognized as

$$\dot{\mathbf{r}}_i\cdot\delta\dot{\mathbf{r}}_i = \frac{\delta(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)}{2} \qquad \textbf{[4.8.3]}$$

The kinetic energy of the ith particle is

$$T_i = \frac{1}{2}m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i \qquad \textbf{[4.8.4]}$$

so that the variation of the kinetic energy of the ith particle becomes

$$\delta T_i = \frac{1}{2}m_i\,\delta(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i) = m_i\dot{\mathbf{r}}_i\cdot\delta\dot{\mathbf{r}}_i \qquad \textbf{[4.8.5]}$$

and we can express Eq. [4.8.2] as

$$m_i\frac{d}{dt}(\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i) = m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + m_i\dot{\mathbf{r}}_i\cdot\delta\dot{\mathbf{r}}_i = m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + \delta T_i \qquad \textbf{[4.8.6]}$$

The variation in the total kinetic energy of the system is

$$\delta T = \sum_{i=1}^{N} \delta T_i = \sum_{i=1}^{N} \frac{1}{2} m_i \, \delta(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) \qquad [4.8.7]$$

Using Eq. [4.8.6], we express D'Alembert's principle as

$$\sum_{i=1}^{N} (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \mathbf{r}_i = 0 = -\delta T + \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) - \delta W \qquad [4.8.8]$$

so that we have an expression for the variation of the kinetic and potential energies

$$\delta T + \delta W = \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) \qquad [4.8.9]$$

Next, we integrate the right side of Eq. [4.8.9] over two points in time, say, t_1 and t_2, thus

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \int_{t_1}^{t_2} \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i)\, dt$$

$$= \int \sum_{i=1}^{N} m_i \, d(\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \Big|_{t_1}^{t_2} \qquad [4.8.10]$$

The term $m_i \dot{\mathbf{r}}_i$ is recognized as the partial derivative of T_i with respect to $\dot{\mathbf{r}}_i$, so that we may write

$$\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \Big|_{t_1}^{t_2} = \sum_{i=1}^{N} \frac{\partial T_i}{\partial \dot{\mathbf{r}}_i} \cdot \delta \mathbf{r}_i \Big|_{t_1}^{t_2} \qquad [4.8.11]$$

which, when introduced back into Eq. [4.8.10], yields

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt - \sum_{i=1}^{N} \frac{\partial T_i}{\partial \dot{\mathbf{r}}_i} \cdot \delta \mathbf{r}_i \Big|_{t_1}^{t_2} = 0 \qquad [4.8.12]$$

This equation is known as *Hamilton's principle (or law) of varying action.* One can put this principle into more general form, by expressing it in terms of generalized coordinates alone. Introducing Eq. [4.4.7] into Eq. [4.8.11], we obtain

$$\sum_{i=1}^{N} \frac{\partial T_i}{\partial \dot{\mathbf{r}}_i} \cdot \delta \mathbf{r}_i \Big|_{t_1}^{t_2} = \sum_{i=1}^{N} \frac{\partial T_i}{\partial \dot{\mathbf{r}}_i} \cdot \sum_{k=1}^{n} \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \delta q_k \Big|_{t_1}^{t_2} = \sum_{k=1}^{n} \frac{\partial T}{\partial \dot{q}_k} \delta q_k \Big|_{t_1}^{t_2} \qquad [4.8.13]$$

Introducing Eq. [4.8.13] into Eq. [4.8.12] we write Hamilton's principle of varying action as

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt - \sum_{k=1}^{n} \frac{\partial T}{\partial \dot{q}_k} \delta q_k \Big|_{t_1}^{t_2} = 0 \qquad [4.8.14]$$

Note that the derivation above does not put any restrictions on the time instances t_1 and t_2.

A special case of Hamilton's principle of varying action is obtained when we consider the variation of $\mathbf{r}_i$ as time is held fixed. We reexamine Fig. 4.12, which is analogous to Fig. B1 in Appendix B. The varied path can take any value within the set of admissible displacements of the system, and it coincides with the true path at the end points. It follows that the variation of the displacement $\delta\mathbf{r}_i$ and of the generalized coordinates have values of zero at $t = t_1$ and $t = t_2$, provided $\mathbf{r}$ is specified at t_1 and t_2. Of interest is the case when $\mathbf{r}_i$ $(i = 1, 2, \ldots, N)$ are specified, which eliminates the integrated term in Hamilton's principle of varying action, resulting in

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0 \qquad \textbf{[4.8.15]}$$

This equation is known as the *extended Hamilton's principle.* Writing the virtual work as $\delta W = \delta W_{nc} - \delta V$, one can express the extended Hamilton's principle also as

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc})\, dt = 0 \qquad \textbf{[4.8.16]}$$

Even though we derived it here for a system of particles, the extended Hamilton's principle is valid both for particles and for rigid or elastic bodies. It is, again, a fundamental principle of mechanics from which the motion of *all* bodies can be described. In this sense, the extended Hamilton's principle is not exactly a derived principle. Rather, it is more like a law of nature, in the same way that Newton's second law is a law of nature. Further, only scalar quantities like work and energy are needed. No acceleration terms need to be calculated to invoke this principle.

Introduce the Lagrangian L such that $L = T - V$. For conservative systems, $\delta W = -\delta V$, and we can write

$$\int_{t_1}^{t_2} \delta L\, dt = 0 \qquad \textbf{[4.8.17]}$$

and Eq. [4.8.17] is referred to as *Hamilton's principle.* This principle was first stated by Lagrange and originally called *Principle of least action.* When the system is holonomic, one can interchange the integration and variation operations, which yields

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad \textbf{[4.8.18]}$$

Hamilton's principle for a holonomic system basically states that among all the paths that a system can take, the actual path followed renders the definite integral $I = \int_{t_1}^{t_2} L\, dt$ stationary. This integral is also known as the *action integral.*

The implementation of the extended Hamilton's principle for finding the equations of motion requires the evaluation of the variations of the kinetic and potential energies. The procedure can become tedious, primarily because of the large number of integrations by parts that one must perform to relate the variations of generalized velocities to the variations of the generalized coordinates. A simpler and more general procedure for deriving the equations of motion for systems with a finite number of degrees of freedom is by means of Lagrange's equations, as we will see in the next section.

The direct use of the extended Hamilton's principle is effective when deriving the equations of motion of deformable bodies, such as for the vibrations of beams, plates, and shells. In such problems, the extended Hamilton's principle yields the equations of motion in the form of partial differential equations with accompanying boundary conditions. We will investigate the dynamics of deformable bodies in Chapter 11. Hamilton's principle is also used in transformation theory and in optimal control theory.

One may wonder why we list two major principles in this section that encompass nonconservative forces when the first, Hamilton's law of varying action, is lengthier and has the appearance of being redundant when compared with the extended principle. The difference between the two principles is in how they treat the time instances t_1 and t_2.

If we view t_1 and t_2 as arbitrary time instances, we obtain the extended Hamilton's principle from Hamilton's law of varying action and the two principles become the same. But if we view t_1 as a point at which we know the values of the generalized coordinates, then we can make use of Eq. [4.8.14] to find the values of the generalized coordinates at time t_2. To do this we do *not* need to derive any equations of motion, just the variation of the Lagrangian and the virtual work. This approach comes in handy in numerical integration, as t_2 can be taken as $t_1 + \Delta$, in which Δ is a small time increment.

Example 4.10

Obtain the equation of motion of the bead problem in Example 4.9 using the extended Hamilton's principle.

Solution

To find the kinetic energy, we need the velocity of the bead. From Example 4.9 we have

$$\mathbf{r} = R\sin\theta\mathbf{b}_2 - R\cos\theta\mathbf{b}_3 \qquad \text{[a]}$$

$$\mathbf{v} = -R\Omega\sin\theta\mathbf{b}_1 + R\dot{\theta}\cos\theta\mathbf{b}_2 + R\dot{\theta}\sin\theta\mathbf{b}_3 \qquad \text{[b]}$$

The kinetic energy is

$$T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m[(\Omega R\sin\theta)^2 + (R\dot{\theta}\cos\theta)^2 + (R\dot{\theta}\sin\theta)^2] = \frac{mR^2}{2}\Omega^2\sin^2\theta + \frac{mR^2}{2}\dot{\theta}^2 \qquad \text{[c]}$$

Using the position of the bead at the bottom of the ring ($\theta = 0$) as the datum, the potential energy becomes

$$V = mgR(1-\cos\theta) \qquad \text{[d]}$$

so that the Lagrangian has the form

$$L = T - V = \frac{mR^2}{2}\Omega^2\sin^2\theta + \frac{mR^2}{2}\dot{\theta}^2 - mgR(1-\cos\theta) \qquad \text{[e]}$$

The variation of the Lagrangian is

$$\delta L = \frac{\partial L}{\partial \theta}\delta\theta + \frac{\partial L}{\partial \dot{\theta}}\delta\dot{\theta} = mR^2\left[\Omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta\right]\delta\theta + mR^2\dot{\theta}\,\delta\dot{\theta} \qquad \textbf{[f]}$$

The second term in this equation is in terms of $\delta\dot{\theta}$. To invoke the extended Hamilton principle, we have to express all the terms in terms of $\delta\theta$. To accomplish this, we integrate this second term by parts and write

$$\int_{t_1}^{t_2} \dot{\theta}\,\delta\dot{\theta}\,dt = \int_{t_1}^{t_2} \dot{\theta}\frac{d}{dt}(\delta\theta)\,dt = \dot{\theta}\,\delta\theta\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\theta}\,\delta\theta\,dt \qquad \textbf{[g]}$$

The integrated term on the right side of Eq. [g] vanishes by virtue of the definition of the variation operation. (The values of the variation at the beginning and end of the path are zero.) The second term, when used with Eq. [f] and the Extended Hamilton's Principle, yields

$$\int_{t_1}^{t_2}\left[-mR^2\ddot{\theta} + mR^2\left(\Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta\right)\right]\delta\theta\,dt = 0 \qquad \textbf{[h]}$$

In order for the equality to hold, the integrand must vanish at all times. Because $\delta\theta$ is arbitrary, for the integrand to be zero the coefficient of $\delta\theta$ must be identically zero. Thus we recognize as the equation of motion

$$\ddot{\theta} + \sin\theta\left(\frac{g}{R} - \Omega^2\cos\theta\right) = 0 \qquad \textbf{[i]}$$

Let us review the operations we carried out. After obtaining the kinetic and potential energies and taking the partial derivatives, we performed an integration by parts on the term $\dot{\theta}\,\delta\dot{\theta}$. We could have done the integration by parts on the general expression $\frac{\partial L}{\partial \dot{\theta}}\,\delta\dot{\theta}$ rather than the corresponding specific term in this problem, $\dot{\theta}\,\delta\dot{\theta}$. The question then arises as to whether, manipulating the extended Hamilton's principle, one can perform the integrations by part in advance and develop a general form for the equations of motion. This is the question we will explore in the next section.

4.9 LAGRANGE'S EQUATIONS

From Hamilton's principle, we derive Lagrange's equations, which present themselves as a convenient way of deriving the equations of motion. The extended Hamilton's principle can be expressed as

$$\int_{t_1}^{t_2}(\delta T - \delta V + \delta W_{nc})\,dt = \int_{t_1}^{t_2}\delta L\,dt + \int_{t_1}^{t_2}\delta W_{nc}\,dt = 0 \qquad \textbf{[4.9.1]}$$

The Lagrangian L can be written in terms of generalized coordinates q_k and generalized velocities $\dot{q}_k$ $(k = 1, 2, \ldots, n)$ as $L = L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$.

The variation of L is

$$\delta L = \sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) \quad \textbf{[4.9.2]}$$

and, using Eqs. [4.5.10] and [4.5.19], the variation of the nonconservative work is written in terms of the generalized forces as

$$\delta W_{nc} = \sum_{k=1}^{n} Q_{knc} \, \delta q_k \quad \textbf{[4.9.3]}$$

Making use of the property that the variation and differentiation (with regard to time) operations can be interchanged, we integrate by parts the second term in Eq. [4.9.2] and obtain

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} (\delta q_k) \, dt = \left. \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right|_{\delta q_k(t_1)}^{\delta q_k(t_2)} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \, dt \quad \textbf{[4.9.4]}$$

The integrated term requires evaluation of δq_k ($k = 1, 2, \ldots, n$) at the beginning and the end of the time intervals. By the definition of the variation, the varied path vanishes at the end points, thus $\delta q_k(t_1) = \delta q_k(t_2) = 0$ for all values of k. Considering this, and introducing Eqs. [4.9.2]–[4.9.4] into the extended Hamilton's principle, we obtain

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) \, dt = \int_{t_1}^{t_2} \sum_{k=1}^{n} \left[-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial L}{\partial q_k} + Q_{knc} \right] \delta q_k \, dt = 0 \quad \textbf{[4.9.5]}$$

For the integral over time to vanish at all times, the integrand must be identically equal to zero, which can be expressed as

$$\sum_{k=1}^{n} \left[-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial L}{\partial q_k} + Q_{knc} \right] \delta q_k = 0 \quad \textbf{[4.9.6]}$$

It should be noted that this equation can be directly obtained from D'Alembert's principle, without using Hamilton's principle. Because of this, Eq. [4.9.6] is sometimes referred to as *Lagrange's form of D'Alembert's principle.*

Consider now a set of independent generalized coordinates. It follows that the only way Eq. [4.9.6] can be equal to zero is if the coefficients of δq_k vanish individually for all values of the index k. Setting the coefficients equal to zero, we obtain *Lagrange's equations* of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_{knc} \qquad k = 1, 2, \ldots, n \quad \textbf{[4.9.7]}$$

Equation [4.9.7] is the most general form of Lagrange's equations. They can also be expressed in terms of the kinetic and potential energies. Noting that the potential energy is not a function of the generalized velocities (except for electromagnetic systems), we write Eq. [4.9.7] as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_{knc} \qquad k = 1, 2, \ldots, n \qquad \textbf{[4.9.8]}$$

This form of Lagrange's equations is preferred by many, as it reduces the possibility of making a sign error when evaluating the partial derivatives. It also is similar to the format Lagrange first presented these equations in 1788. Under certain circumstances it is more convenient to write Lagrange's equations in terms of the kinetic energy alone, in the form of

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \qquad \textbf{[4.9.9]}$$

where the values of Q_k contain contributions from the conservative as well as non-conservative forces. The principle of virtual work given by Eq. [4.6.4], is a special case of Lagrange's equations. In the static case, the first two terms in Eq. [4.9.8] vanish.

For a holonomic conservative system, one can use Eq. [4.8.15] directly in conjunction with the Euler-Lagrange equation in Appendix B to derive Lagrange's equations. The order of variation and integration can be exchanged, and one seeks the stationary values of the integral $I = \int_{t_1}^{t_2} L\, dt$, leading to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \textbf{[4.9.10]}$$

Lagrange's equations can conveniently be expressed in column vector format. Introducing the n-dimensional generalized coordinate and generalized force vectors

$$\{q\} = [q_1 \quad q_2 \quad \ldots \quad q_n]^T \qquad \{Q_{nc}\} = [Q_{1nc} \quad Q_{2nc} \quad \ldots \quad Q_{nnc}]^T \qquad \textbf{[4.9.11]}$$

we can write Lagrange's equations as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\{\dot{q}\}}\right) - \frac{\partial L}{\partial\{q\}} = \{Q_{nc}\}^T \qquad \textbf{[4.9.12]}$$

Let us now compare the steps involved in obtaining the equations of motion using Lagrange's equations and using the Newtonian approach. When using Newton's second law, we

1. Isolate the different bodies involved.
2. Select a coordinate system and draw free-body diagrams.
3. Relate the sum of forces and sum of moments to the translational and angular accelerations.
4. Use kinematics to express the accelerations in terms of translational and angular parameters.
5. Eliminate the constraint and reaction forces and derive the equations of motion.

When using the Lagrangian approach, we

1. Determine the number of degrees of freedom and select a set of independent generalized coordinates. The free-body diagram is a useful tool for this.
2. Use the kinematical relations to find the velocities and virtual displacements involved.
3. Identify the forces that are conservative and those that are not.
4. Write the kinetic and potential energies, as well as the virtual work.
5. Apply Lagrange's equations.

There are two distinct differences between the two approaches. The first difference is in the order of the steps involved: In the Newtonian approach, one first writes the force and moment balances for all bodies separately and then uses kinematical relations and the constraint forces to reduce the number of equations. In the Lagrangian approach, one considers the constraints and kinematics of the problem first. Then, the equations of motion are written, one for each degree of freedom. The bulk of the work involved in Lagrangian mechanics is to find a proper set of generalized coordinates and to express the kinematics. Once this is done, the rest is straightforward.

The second difference is that the Lagrangian approach uses velocities and scalar quantities, whereas the Newtonian approach uses accelerations and vector quantities. Dealing with velocities involves considerably less algebra than dealing with accelerations.

It may appear, from the above discussion, that Lagrange's equations should be preferable to the Newtonian approach at all times; but this is not so. By eliminating the constraint forces from the formulation, the Lagrangian approach does not calculate the amplitudes of these forces. While this may be acceptable for classroom examples, it certainly is not in many real-life applications, where one must know the amplitudes of the reaction and other contact forces acting on a body. Furthermore, for certain geometries a Newtonian approach is more suitable. The best way to determine which approach is most suited to one's needs is by gaining experience in solving mechanics problems. In many cases, looking at a problem from *both* a Lagrangian and Newtonian point of view increases the physical insight and makes it easier to understand the characteristics of the system.

We should add here that the historical development of analytical mechanics did not follow the sequence in which it is presented in this chapter. Lagrange's equations were derived before the extended Hamilton's principle, and they were derived for conservative systems only. It was Hamilton, born after Lagrange, who put together the developments in variational mechanics and Lagrange's equations to develop a general scalar principle from which all the equations of motion can be derived.

Example 4.11

For the system in Fig. 4.20, find the equations of motion using Lagrange's equations. Assume that the spring and dashpot deflect only horizontally and that the force F is always applied horizontally.

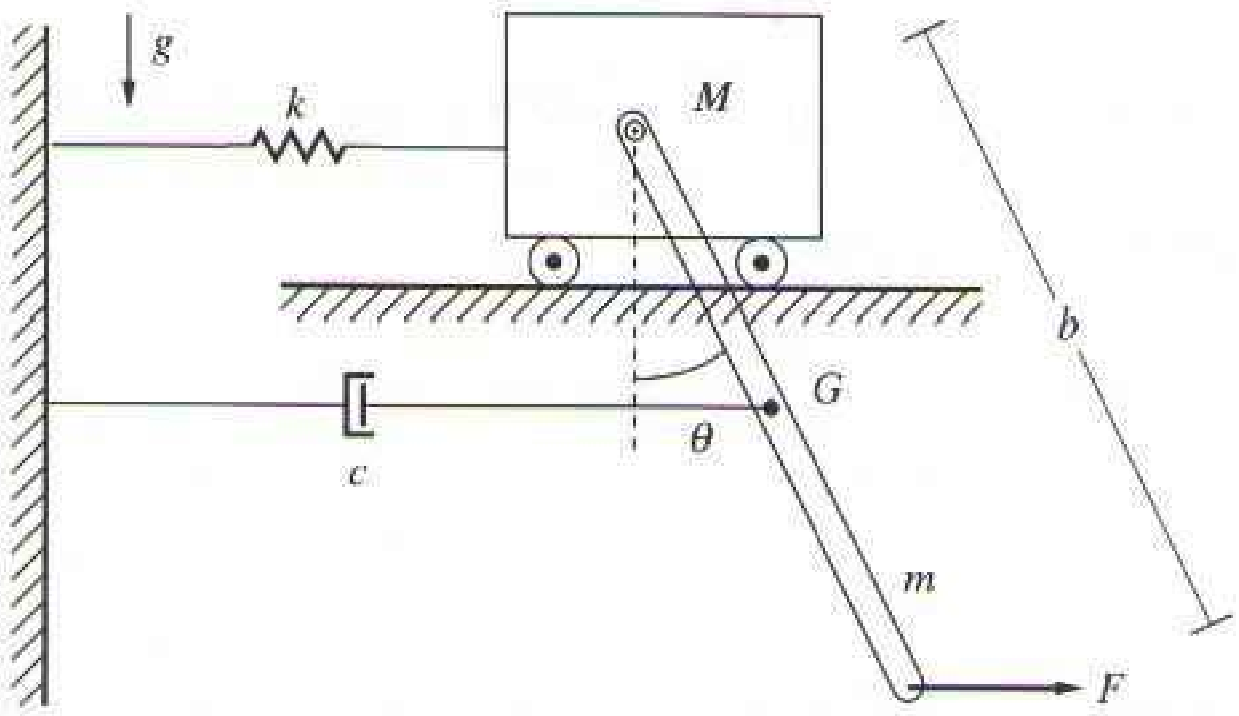

Figure 4.20

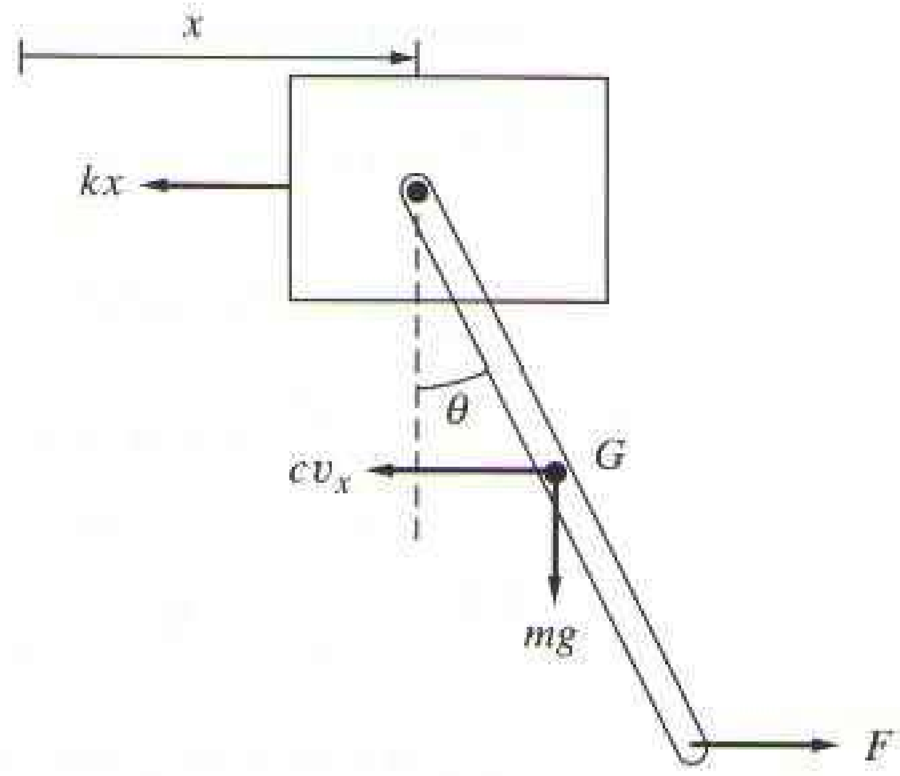

Figure 4.21 Free body diagram

Solution

This is a two degree of freedom system, and we select the generalized coordinates as the displacement of the mass x and rotation of the bar θ. The free-body diagram of the entire system is shown in Fig. 4.21. The kinetic energy of the cart is $T_{\text{cart}} = \frac{1}{2}M\dot{x}^2$. The kinetic energy of the bar is due to the translation and rotation and can be expressed as

$$T_{\text{bar}} = \frac{1}{2}I_G\dot{\theta}^2 + \frac{1}{2}m(v_x^2 + v_y^2) \qquad \textbf{[a]}$$

where I_G is the mass moment of inertia about the center of mass, $I_G = mb^2/12$, and v_x and v_y are the velocities of the center of mass of the bar, found as

$$v_x = \frac{d}{dt}x_G = \frac{d}{dt}\left(x + \frac{b}{2}\sin\theta\right) = \dot{x} + \frac{b}{2}\dot{\theta}\cos\theta$$

$$v_y = \frac{d}{dt}y_G = \frac{d}{dt}\left(-\frac{b}{2}\cos\theta\right) = \frac{b}{2}\dot{\theta}\sin\theta \qquad \textbf{[b]}$$

The total kinetic energy is

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{24}mb^2\dot{\theta}^2 + \frac{1}{2}m\left[\left(\dot{x} + \frac{b}{2}\dot{\theta}\cos\theta\right)^2 + \left(\frac{b}{2}\dot{\theta}\sin\theta\right)^2\right]$$

$$= \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}mb\dot{\theta}\dot{x}\cos\theta + \frac{1}{6}mb^2\dot{\theta}^2 \qquad \textbf{[c]}$$

The potential energy is due to the deflection of the spring and the vertical movement of the center of mass of the bar, written

$$V = \frac{1}{2}kx^2 - mg\frac{b}{2}\cos\theta \qquad \textbf{[d]}$$

The virtual work of the nonconservative forces is due to the external force F and the dashpot, so

$$\delta W_{nc} = F\,\delta r - cv_x\,\delta x_G = F\,\delta(x + b\sin\theta) - c\left[\dot{x} + \frac{b}{2}\dot{\theta}\cos\theta\right]\left(\delta x + \frac{b}{2}\cos\theta\,\delta\theta\right)$$

$$= F\,\delta x + Fb\cos\theta\,\delta\theta - c\dot{x}\,\delta x - \frac{1}{2}cb\dot{\theta}\cos\theta\,\delta x - \frac{1}{2}cb\dot{x}\cos\theta\,\delta\theta - \frac{1}{4}cb^2\dot{\theta}\cos^2\theta\,\delta\theta \qquad \textbf{[e]}$$

from which we recognize the generalized forces as

$$Q_x = F - c\dot{x} - \frac{1}{2}cb\cos\theta \qquad Q_\theta = Fb\cos\theta - \frac{1}{4}cb^2\dot{\theta}\cos^2\theta - \frac{1}{2}cb\dot{x}\cos\theta \qquad \textbf{[f]}$$

Taking the appropriate derivatives, we obtain

$$\frac{\partial T}{\partial \dot{x}} = (M+m)\dot{x} + \frac{1}{2}mb\dot{\theta}\cos\theta \qquad \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2}mb\dot{x}\cos\theta + \frac{1}{3}mb^2\dot{\theta}$$

$$-\frac{\partial T}{\partial x} = 0 \qquad \frac{\partial V}{\partial x} = kx \qquad -\frac{\partial T}{\partial \theta} = \frac{1}{2}mb\dot{\theta}\dot{x}\sin\theta \qquad \frac{\partial V}{\partial \theta} = \frac{1}{2}mgb\sin\theta \qquad \textbf{[g]}$$

Substituting the above values into Lagrange's equations we obtain the equations of motion as

$$(M+m)\ddot{x} + \frac{1}{2}mb\ddot{\theta}\cos\theta - \frac{1}{2}mb\dot{\theta}^2\sin\theta + c\dot{x} + \frac{1}{2}cb\dot{\theta}\cos\theta + kx = F$$

$$\frac{1}{3}mb^2\ddot{\theta} + \frac{1}{2}mb\ddot{x}\cos\theta + \frac{1}{2}cb\dot{x}\cos\theta + \frac{1}{4}cb^2\dot{\theta}\cos^2\theta + \frac{1}{2}mgb\sin\theta = Fb\cos\theta \qquad \textbf{[h]}$$

Example 4.12

Figure 4.22 shows a collar of mass m sliding outside a long, slender rod of mass M and length L. The coefficient of friction between the rod and collar is μ. There is a force F acting at the tip of the rod. Find the equations of motion.

Solution

We will solve this problem as a two degree of freedom unconstrained system. Polar coordinates are suitable as generalized coordinates. The free-body diagrams are given in Fig. 4.23. There are four external forces: two gravity forces, which we will account for in the potential energy, the friction force, and the force at the tip.

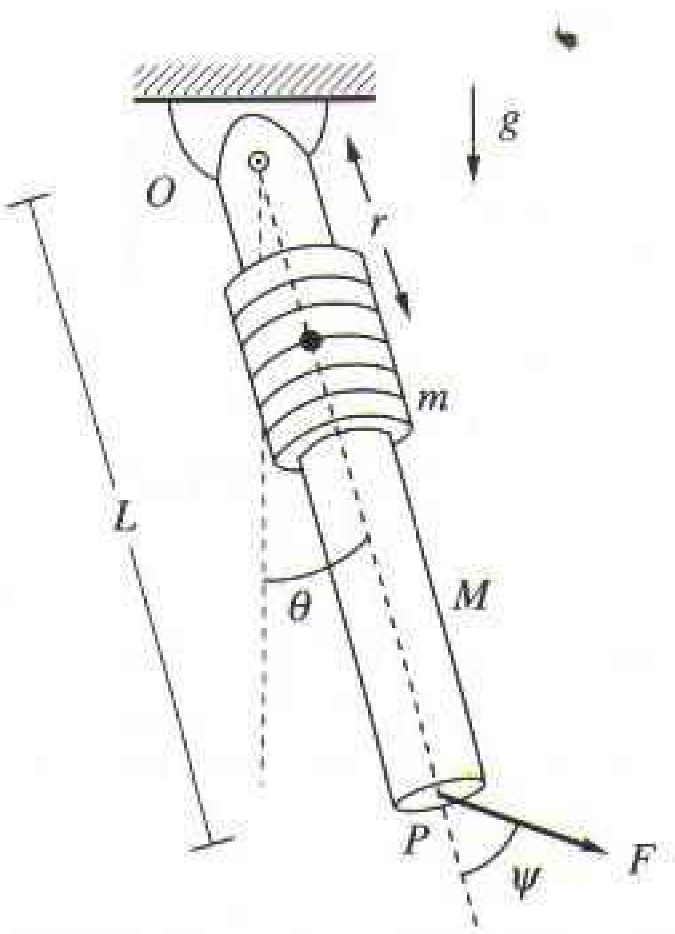

Figure 4.22 Collar sliding on a rod

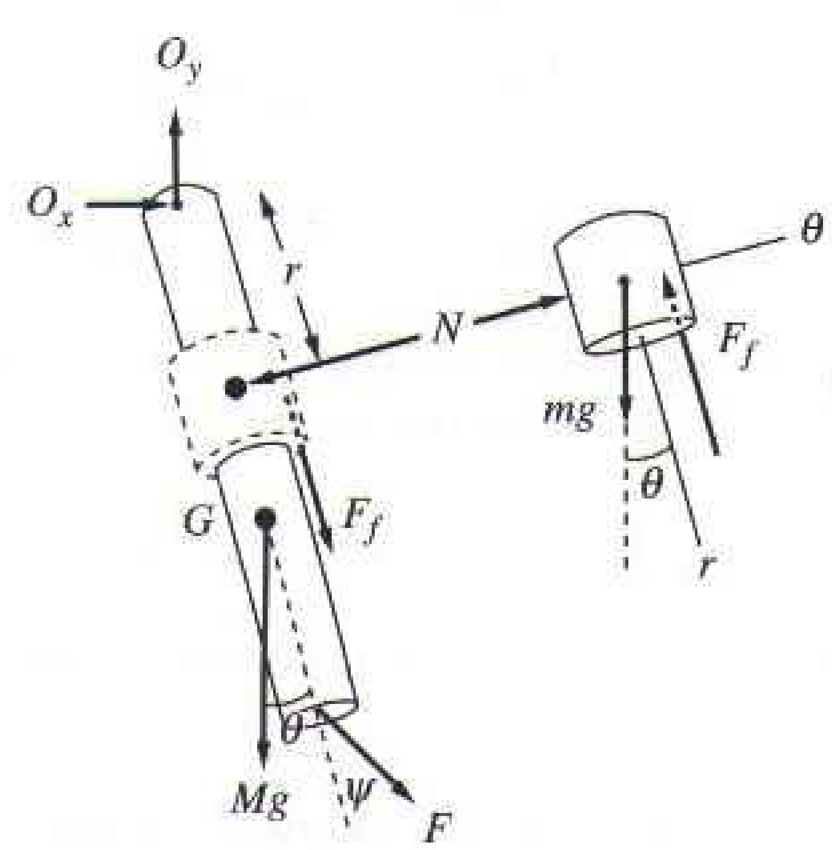

Figure 4.23 Free-body diagram

The position and velocity of the collar are

$$\mathbf{r} = r\mathbf{e}_r \qquad \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{a}$$

The virtual work associated with the two external forces can be written as

$$\delta W = \mathbf{F}\cdot\delta\mathbf{r}_P + \mathbf{F}_f\cdot\delta\mathbf{r} \tag{b}$$

in which

$$\mathbf{F} = F\cos\psi\mathbf{e}_r + F\sin\psi\mathbf{e}_\theta \quad \mathbf{F}_f = -F_f\operatorname{sign}(\dot{r})\mathbf{e}_r \quad \delta\mathbf{r} = \delta r\mathbf{e}_r + r\,\delta\theta\mathbf{e}_\theta \quad \delta\mathbf{r}_P = L\,\delta\theta\mathbf{e}_\theta \tag{c}$$

so that the virtual work becomes

$$\delta W = -F_f\operatorname{sign}(\dot{r})\,\delta r + FL\sin\psi\,\delta\theta = Q_r\,\delta r + Q_\theta\,\delta\theta \tag{d}$$

with $Q_r = -F_f\operatorname{sign}(\dot{r})$ and $Q_\theta = FL\sin\psi$ as the generalized forces due to the nonconservative forces.

The kinetic energy is

$$T = T_{\text{rod}} + T_{\text{collar}} = \frac{1}{6}ML^2\dot{\theta}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{e}$$

and the potential energy is

$$V = -mgr\cos\theta - Mg\frac{L}{2}\cos\theta \tag{f}$$

Application of Lagrange's equations yields the equations of motion as

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = -F_f\operatorname{sign}(\dot{r}) \tag{g}$$

$$\left(\frac{1}{3}ML^2 + mr^2\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + \left(mr + \frac{1}{2}ML\right)g\sin\theta = FL\sin\psi \tag{h}$$

The friction force is related to the normal force N between the collar and rod by $F_f = \mu N$. However, at this point we do not know what the normal force is. To find the normal force, we need to go to a Newtonian analysis. Reconsidering the free-body diagram and summing forces along the transverse direction, we obtain

$$\sum F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = N - mg\sin\theta \tag{i}$$

from which we obtain the magnitude of the normal force as

$$N = m(r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta) \tag{j}$$

We can eliminate the normal force from the equations of motion by introducing Eq. [j] into Eq. [g]. Note that the friction force is always a positive quantity, as it is proportional to the magnitude of the normal force. The expression involving N in Eq. [j] can lead to both positive and negative values. Therefore, we express the friction force as

$$F_f = \mu|N| = \mu m|r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta| \tag{k}$$

and use Eq. [k] in the equations of motion.

The preceding example illustrates the problems that one encounters when dealing with problems involving friction. As stated earlier, friction is not a constraint force, but its magnitude depends on a constraint force. If we select a set of unconstrained generalized coordinates to describe the motion, as we did in this example, we cannot obtain the magnitudes of the friction force without an additional Newtonian analysis. In the next section, we will see an analytical approach that calculates magnitudes of constraint forces.

4.10 LAGRANGE'S EQUATIONS FOR CONSTRAINED SYSTEMS

The formulation of Lagrange's equations in the previous section was for unconstrained systems and for constrained systems where the generalized coordinates are selected such that all constraints are accounted for and the surplus coordinates eliminated. This approach is not feasible under a number of circumstances:

1. When the constraints are nonholonomic. Because nonholonomic constraints involve velocity expressions that cannot be integrated to displacement expressions, one cannot find a set of unconstrained generalized coordinates.
2. When the constraints are holonomic and one cannot eliminate the surplus coordinates easily, for one of the following reasons:
   a. The constraint equation is complicated.
   b. Finding the transformations that lead to unconstrained equations makes the equations of motion very complicated.
   c. Some of the forces acting on the system are functions of constraint forces.
3. When the constraints are holonomic but one does not want to eliminate the surplus coordinates from the formulation, usually because of the need to know the amplitudes of the reaction forces.

Consider a system originally of n degrees of freedom, to which m constraints are applied. For the most general case, we express the constraints in velocity form as

$$\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0 \qquad j = 1, 2, \ldots, m \tag{4.10.1}$$

whose variation is

$$\sum_{k=1}^{n} a_{jk}\,\delta q_k = 0 \tag{4.10.2}$$

Multiplying Eq. [4.10.2] by the Lagrange multipliers λ_j $(j = 1, 2, \ldots, m)$ and introducing these constraints to the extended Hamilton's principle, we obtain

$$\int_{t_1}^{t_2} \delta L\,dt + \int_{t_1}^{t_2} \delta W_{nc}\,dt - \int_{t_1}^{t_2} \sum_{j=1}^{m}\sum_{k=1}^{n} \lambda_j a_{jk}\,\delta q_k\,dt = 0 \tag{4.10.3}$$

When the constraints are holonomic, the coordinates $q_1, q_2, \ldots, q_n$ no longer constitute a set of independent generalized coordinates. They are now *constrained* *generalized coordinates.* When the constraints are nonholonomic, only the generalized velocities are constrained, while the generalized coordinates are still independent. In both cases, the variations of the generalized coordinates are constrained. Following the same procedure as when deriving Lagrange's equations for the unconstrained case, we take the appropriate partial derivatives and perform the integrations by parts to obtain

$$\sum_{k=1}^{n}\int_{t_1}^{t_2}\left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)+\frac{\partial L}{\partial q_k}+Q_{knc}-\sum_{j=1}^{m}\lambda_j a_{jk}\right]\delta q_k\,dt = 0 \quad \textbf{[4.10.4]}$$

As in the static case, we select the Lagrange multipliers λ_j such that the coefficients of δq_k $(k = 1, 2, \ldots, n)$ vanish, which leads to a modified form of Lagrange's equations, written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}+\sum_{j=1}^{m}\lambda_j a_{jk} = Q_{knc} \qquad k = 1, 2, \ldots, n \quad \textbf{[4.10.5]}$$

where $\lambda_j a_{jk}$ are the *generalized constraint forces.* They have the same units as the generalized forces (which do not necessarily have the units of force). In column vector notation, Eq. [4.10.5] is expressed as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \{\dot{q}\}}\right)-\frac{\partial L}{\partial \{q\}}+\{\lambda\}^T[a] = \{Q_{nc}\}^T \quad \textbf{[4.10.6]}$$

in which $[a]$ is a matrix of order $m \times n$ whose entries are a_{jk} and $\{\lambda\}$ is a column vector of order m that contains the Lagrange multipliers.

After obtaining the equations of motion, one has two courses of action for finding a solution. The first is to eliminate the Lagrange multipliers from the equations of motion and obtain a set of $n - m$ unconstrained equations. One accomplishes this by algebraic manipulation of the equations of motion. Many times, such an approach results in complicated expressions.

The second course of action is to take the n equations of motion in Eq. [4.10.5] and the m constraint relations in Eq. [4.10.1] and then to solve them together for the $n + m = p + 2m$ unknowns $q_1, q_2, \ldots, q_n, \lambda_1, \lambda_2, \ldots, \lambda_m$. The resulting $n + m$ equations are not a set of differential equations, as there is no derivative of the Lagrange multipliers involved. Such equations are known as *differential-algebraic* equations. Their analysis requires a different treatment than that for differential equations.

When the constraints are holonomic and expressed in the configuration form [4.3.2], one can add them to the extended Hamilton's principle by

$$\int_{t_1}^{t_2}\delta L\,dt+\int_{t_1}^{t_2}\delta W_{nc}\,dt-\int_{t_1}^{t_2}\sum_{j=1}^{m}\lambda_j\,\delta c_j(q_1, q_2, \ldots, q_n) = 0 \quad \textbf{[4.10.7]}$$

and obtain the contribution of the constraint by replacing a_{jk} in Eq. [4.10.5] with $\partial c_j/\partial q_k$. Or, we can add them directly to the Lagrangian as

$$\hat{L} = T - V - \sum_{j=1}^{m}\lambda_j c_j(q_1, q_2, \ldots, q_n) \quad \textbf{[4.10.8]}$$

When the objective is to obtain the amplitude of a constraint force, an analytical approach that can be used is the *constraint relaxation* method. This method is mathematically equivalent to the Lagrange multiplier approach. However, it is more intuitive and it is particularly useful when dealing with holonomic constraints expressed in configuration form. Following is a description of the method.

We relax the constraint from the formulation and represent the effects of the constraint by a constraint force. Then, we write the Lagrangian and virtual work. The constraint force enters the formulation via the virtual work. We invoke Lagrange's equations and obtain the equations of motion. We next impose the constraint, which enables us to calculate the magnitude of the constraint force.

Example 4.13

Consider Example 4.12, in which a collar of mass m is sliding on a rod of mass M and length L. The coefficient of friction between the rod and collar is μ. Obtain the equations of motion using constrained generalized coordinates and find the value of the normal force N.

Solution

To describe this system in terms of constrained generalized coordinates, consider the rod and the collar separately. We express the motion of the collar using polar coordinates, r and θ, as in Example 4.12. To express the motion of the rod, we introduce another angle, ϕ. The constraint equation is

$$\theta - \phi = 0 \qquad \textbf{[a]}$$

The kinetic and potential energy has the same form as in Example 4.12. We write them here in terms of the constrained generalized coordinates as

$$T = \frac{1}{6}ML^2\dot{\phi}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad V = -mgr\cos\theta - Mg\frac{L}{2}\cos\phi \qquad \textbf{[b]}$$

The normal force N acts in the transverse direction and it contributes to the virtual work. Considering that the velocity of the collar in the transverse direction is $v_\theta = r\dot{\theta}\mathbf{e}_\theta$, we write the virtual work expression as

$$\delta W = -F_f \operatorname{sign}(\dot{r})\,\delta r + FL\sin\psi\,\delta\theta + Nr(\delta\theta - \delta\phi) \qquad \textbf{[c]}$$

We obtain the Lagrange's equations as

$$\text{For } r \rightarrow m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = -F_f \operatorname{sign}(\dot{r}) \qquad \textbf{[d]}$$

$$\text{For } \theta \rightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = Nr \qquad \textbf{[e]}$$

$$\text{For } \phi \rightarrow \frac{1}{3}ML^2\ddot{\phi} + \frac{1}{2}MgL\sin\phi = -Nr + FL\sin\psi \qquad \textbf{[f]}$$

These equations have to be solved together with Eq. [a]. Equation [d] is the same as Eq. [g] in Example 4.12, and if we add Eqs. [e] and [f] and use the constraint equation [a] we eliminate the normal force and obtain Eq. [h] in Example 4.12. From Eq. [e], we find the normal force as

$$N = mr\ddot{\theta} + 2m\dot{r}\dot{\theta} + mg\cos\theta \qquad \textbf{[g]}$$

which is the same value obtained in Example 4.12. The friction force is given in Eq. [k] of Example 4.12.

The difference between the approach here and the approach in Example 4.12 is that here we calculated the normal force directly from the Lagrange's equations, while in Example 4.12 we conducted a force balance in addition to the Lagrange's equations.

Example 4.14

Consider the vehicle in Fig. 4.8. Given that the velocity of point A is along the line of symmetry of the vehicle, derive the equations of motion. Gravity acts perpendicular to the plane of motion.

Solution

We denote the coordinates of the center of mass by X and Y and select the generalized coordinates as X, Y, and θ. The kinetic energy of the vehicle is

$$T = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}I_G\dot{\theta}^2 \quad \textbf{[a]}$$

where m is the mass and I_G is the centroidal mass moment of inertia. There is no potential energy, and the virtual work expression involves the two forces F_C and F_D. We can find the virtual work conveniently by calculating the velocities of points C and D. Defining a coordinate system xy attached to the vehicle, we write the velocities of points G and A as

$$\mathbf{v}_G = \dot{X}\mathbf{I} + \dot{Y}\mathbf{J} = (\dot{X}\cos\theta + \dot{Y}\sin\theta)\mathbf{i} + (-\dot{X}\sin\theta + \dot{Y}\cos\theta)\mathbf{j} \quad \textbf{[b]}$$

$$\mathbf{v}_A = v_G + \dot{\theta}\mathbf{k} \times -L\mathbf{i} = (\dot{X}\cos\theta + \dot{Y}\sin\theta)\mathbf{i} + (-\dot{X}\sin\theta + \dot{Y}\cos\theta - L\dot{\theta})\mathbf{j} \quad \textbf{[c]}$$

The constraint is defined as

$$f = \mathbf{v}_A \cdot \mathbf{j} = -\dot{X}\sin\theta + \dot{Y}\cos\theta - L\dot{\theta} = 0 \quad \textbf{[d]}$$

thus the velocity of A reduces to

$$\mathbf{v}_A = (\dot{X}\cos\theta + \dot{Y}\sin\theta)\mathbf{i} \quad \textbf{[e]}$$

and the variation of the constraint becomes

$$\delta f = \sin\theta\,\delta X - \cos\theta\,\delta Y + L\,\delta\theta = 0 \quad \textbf{[f]}$$

Hence, the velocities of C and D become

$$\mathbf{v}_C = \mathbf{v}_A + \dot{\theta}\mathbf{k} \times h\mathbf{j} = (\dot{X}\cos\theta + \dot{Y}\sin\theta - h\dot{\theta})\mathbf{i} \quad \textbf{[g]}$$

$$\mathbf{v}_D = \mathbf{v}_A + \dot{\theta}\mathbf{k} \times (-h\mathbf{j}) = (\dot{X}\cos\theta + \sin\theta + h\dot{\theta})\mathbf{i} \quad \textbf{[h]}$$

The external forces are $\mathbf{F}_C = F_C\mathbf{i}$, $\mathbf{F}_D = F_D\mathbf{i}$, so the virtual work expression becomes

$$\begin{aligned}\delta\hat{W} &= \mathbf{F}_C \cdot \delta\mathbf{r}_C + \mathbf{F}_D \cdot \delta\mathbf{r}_D + \lambda\,\delta f \\ &= (F_C + F_D)\cos\theta\,\delta X + (F_C + F_D)\sin\theta\,\delta Y + (F_D - F_C)h\,\delta\theta \\ &\quad + \lambda(\delta X\sin\theta - \delta Y\cos\theta + L\,\delta\theta)\end{aligned} \quad \textbf{[i]}$$

The physical interpretation of the Lagrange multiplier is that it is the resultant of all forces that keep $\mathbf{v}_A$ along the x axis, and it acts in a direction perpendicular to this axis. Introducing Eqs. [a] and [i] into Lagrange's equations, we obtain the equations of motion as

$$m\ddot{X} = (F_C + F_D)\cos\theta + \lambda\sin\theta \qquad [j]$$

$$m\ddot{Y} = (F_C + F_D)\sin\theta - \lambda\cos\theta \qquad [k]$$

$$I_G\ddot{\theta} = (F_D - F_C)h + L\lambda \qquad [l]$$

Note that while deriving the equations of motion we did not introduce Eq. [d] directly into the expression for the kinetic energy, thus eliminating one of the generalized coordinates from the outset. Had we done so, we would have eliminated the contribution due to the variation of that coordinate and ended up with an incorrect representation. This procedure is crucial to the treatment of nonholonomic constraints.

Even if we eliminated one of the generalized velocities and the Lagrange multiplier, writing the equations of motion in terms of Y and θ (or X and θ) would not give the most meaningful description of the motion. A quantity critical to the understanding of the motion is the speed of point A. If the equations of motion can be expressed in terms of that speed, one gets a clearer picture of the nature of the motion. One can introduce v_A to Eqs. [j]–[l] with a substitution.

Indeed, if we multiply Eq. [j] by $\cos\theta$ and Eq. [k] by $\sin\theta$ and add the two we obtain

$$m(\ddot{X}\cos\theta + \ddot{Y}\sin\theta) = F_C + F_D \qquad [m]$$

Recalling from Eq. [e] that $v_A = (\dot{X}\cos\theta + \dot{Y}\sin\theta)$, differentiating this expression we obtain

$$\dot{v}_A = \ddot{X}\cos\theta + \ddot{Y}\sin\theta + \dot{\theta}(-\dot{X}\sin\theta + \dot{Y}\cos\theta) \qquad [n]$$

Introducing Eq. [d] to Eq. [n] we can write

$$\ddot{X}\cos\theta + \ddot{Y}\sin\theta = \dot{v}_A - L\dot{\theta}^2 \qquad [o]$$

so that Eq. [m] can be written as

$$m(\dot{v}_A - L\dot{\theta}^2) = F_C + F_D \qquad [p]$$

which is recognized as the force balance along the x direction.

We next find an expression for the Lagrange multiplier λ and introduce it to Eq. [e]. To this end, we multiply Eq. [j] with $\sin\theta$ and Eq. [k] by $-\cos\theta$ and add the two, with the result

$$m(\ddot{X}\sin\theta - \ddot{Y}\cos\theta) = \lambda \qquad [q]$$

We can introduce this relationship to Eq. [l], but a more meaningful expression can be generated if we consider Eq. [d] and differentiate it

$$\ddot{X}\sin\theta - \ddot{Y}\cos\theta + \dot{\theta}(\dot{X}\cos\theta + \dot{Y}\sin\theta) = -L\ddot{\theta} \qquad [r]$$

Considering Eq. [q], we express the Lagrange multiplier as

$$\lambda = -mv_A\dot{\theta} - mL\ddot{\theta} \qquad [s]$$

Introducing this equation into Eq. [l] we obtain

$$(I_G + mL^2)\ddot{\theta} + mLv_A\dot{\theta} = (F_D - F_C)h \qquad [t]$$

which we recognize as the moment balance about point A. Equations [p] and [t] are the two independent equations of motion of the vehicle.

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HOMEWORK EXERCISES

SECTION 4.3

1. The four-bar linkage in Fig. 4.24 is a single degree of freedom system. Show that this is so by separating the mechanism into its three components and by writing the constraint equations that relate the configurations of the links.

2. A bead slides up a spiral of constant radius R and height h, as shown in Fig. 4.25. It takes the bead six full turns to reach the top. Express the characteristics of the path of the bead as a constraint relation.

3. A particle slides inside a smooth paraboloid of revolution described by $z = r^2/b$, as shown in Fig. 4.26. Using cylindrical coordinates, find an expression for the constraint force on the particle.

4. The radar tracking of a moving vehicle by another moving vehicle is a common problem. Consider the two vehicles A and B in Fig. 4.27. The orientation of vehicle A must always be toward vehicle B. Express the constraint relation between the velocities and distance between the two vehicles and determine whether this is a holonomic constraint or not.

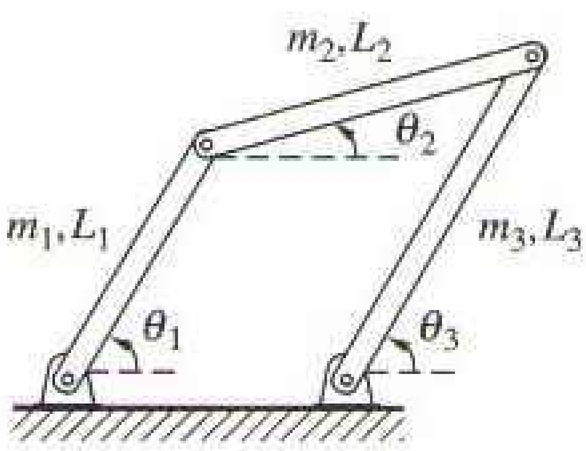

Figure 4.24 Four-bar linkage

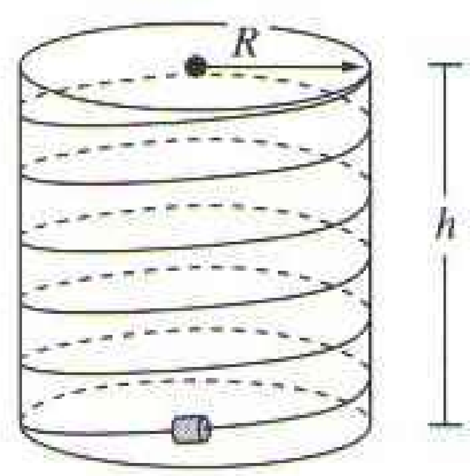

Figure 4.25

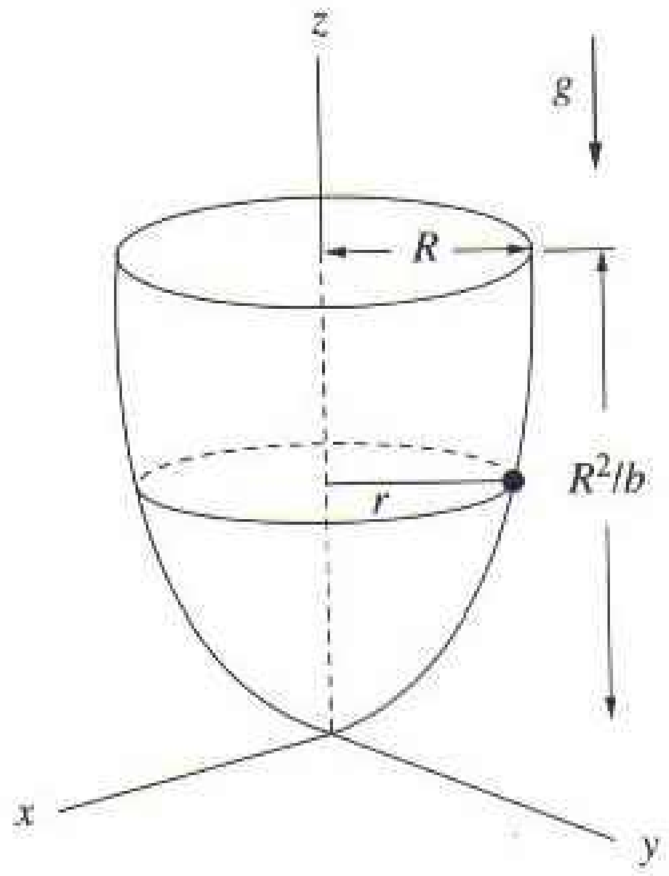

Figure 4.26

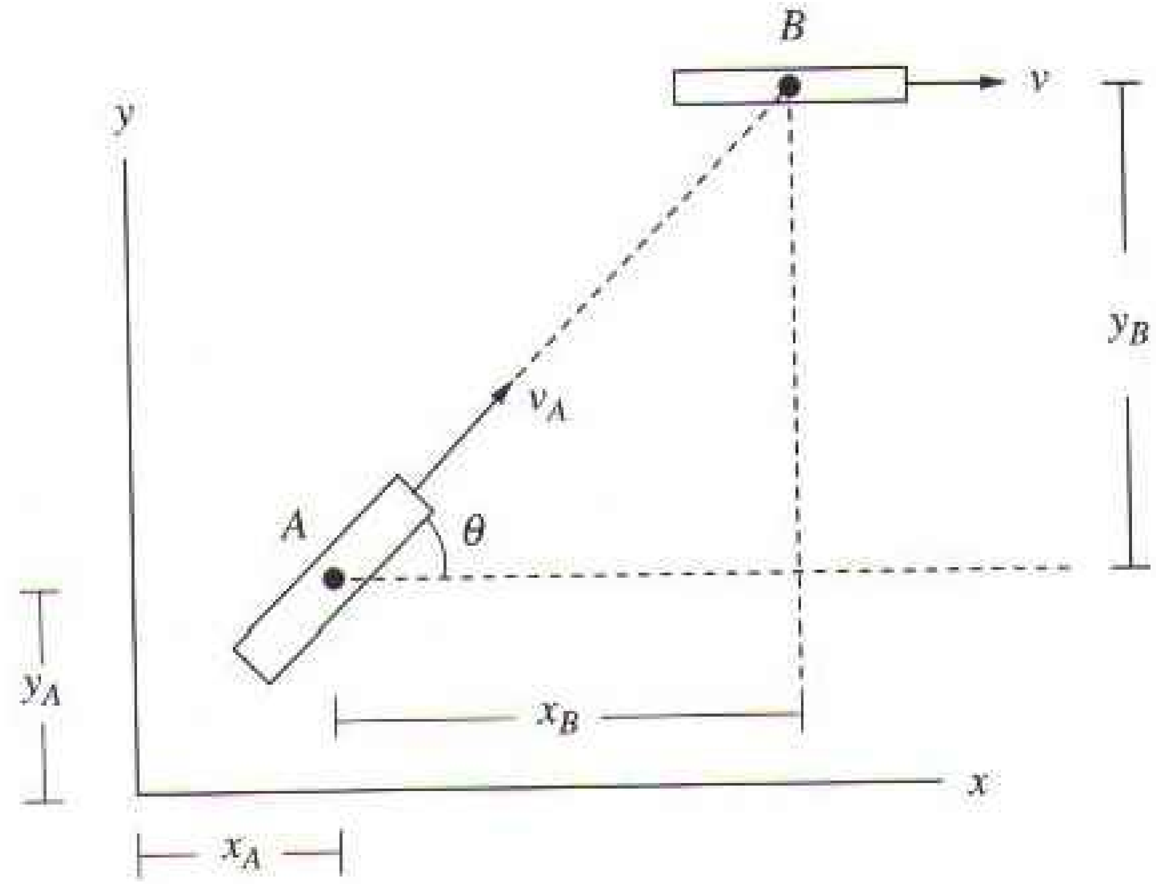

Figure 4.27

5. Consider the double pendulum in Fig. 4.3. It is desired to have the velocity of the tip of the pendulum point toward the pinned end O. Express this condition as a constraint and determine whether the constraint is holonomic or not.

SECTION 4.4

6. Consider a particle moving along a path and the description of motion by path variables. Express the force keeping the particle moving along the path in terms of its components in the tangential, normal, and binormal directions and evaluate the virtual work expression. Identify which of these forces are constraint forces and verify Eq. [4.4.14].
7. Find the virtual displacement of point P in Fig. 4.28. The mass is suspended from an arm which is attached to a rotating column. The pendulum swings in the plane generated by the column and arm.

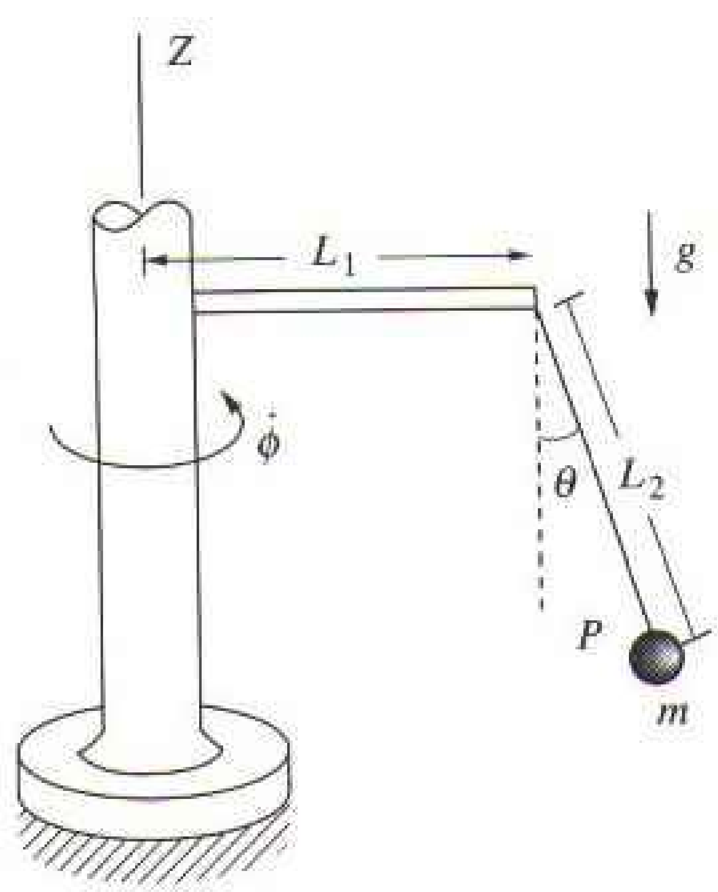

Figure 4.28

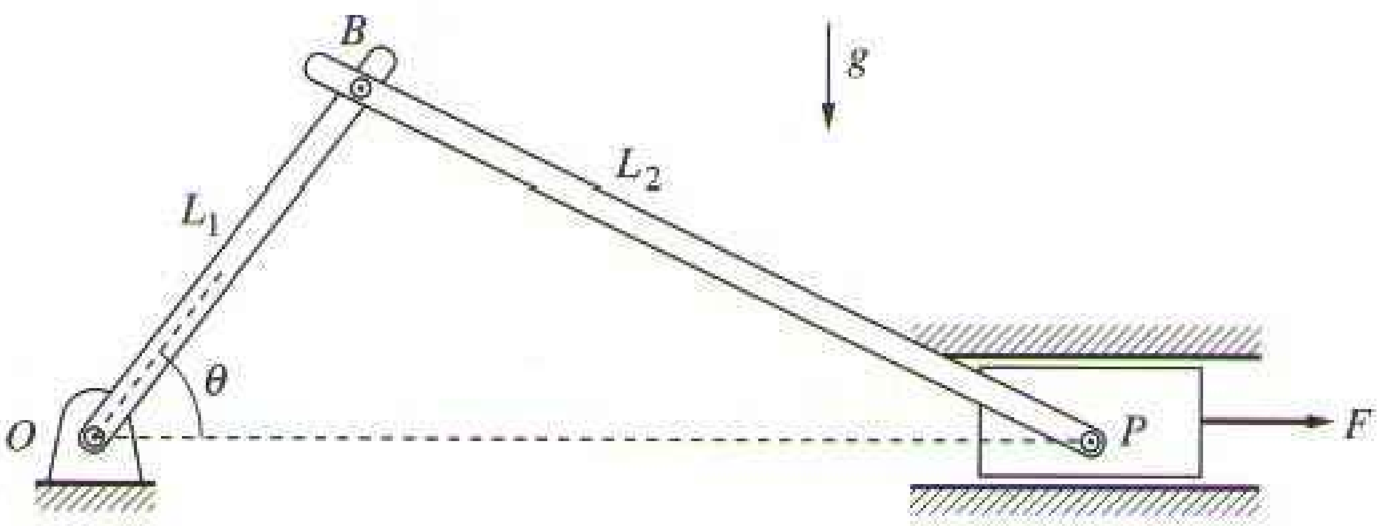

Figure 4.29 Slider-crank mechanism

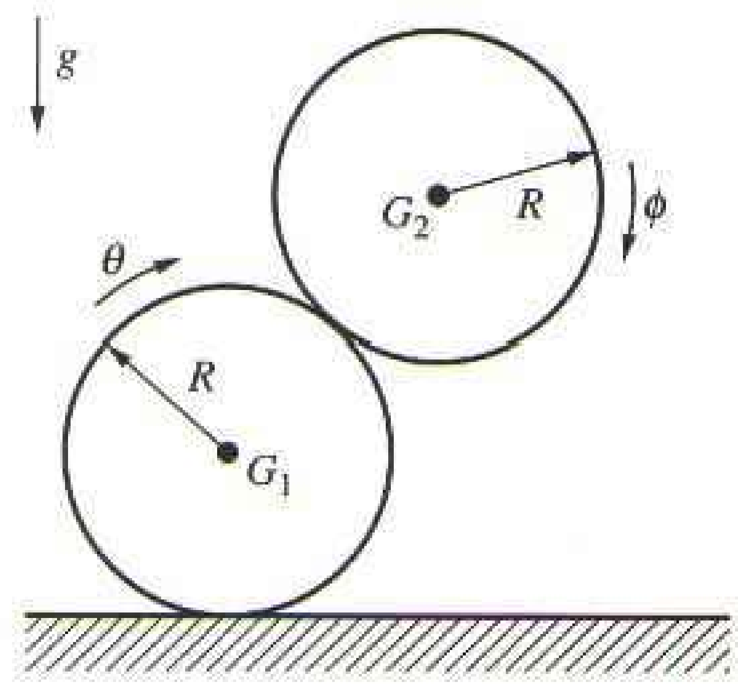

Figure 4.30

8. Express the virtual displacement of the slider in the slider-crank mechanism shown in Fig. 4.29 using (*a*) the relative velocity relations, and (*b*) the analytical expressions.
9. A uniform solid cylinder of radius R rolls without slip on a horizontal plane and an identical cylinder rolls without slip on it (Fig. 4.30). Find the virtual displacements of the centers of the cylinders.
10. Consider Fig. 4.2 and the case when the cord is getting pulled down by an external force, such that the length of the cord varies by $L(t) = L_0 e^{-0.2t}$. Find the virtual displacement of the mass.

SECTION 4.5

11. Find the generalized force associated with the system in Fig. 4.29.
12. The spherical pendulum of mass m shown in Fig. 4.2 has its length being reduced by a force F, according to the relationship $L(t) = L_0 - bt$, where L_0 is the initial length and b is a constant. Calculate the generalized forces using spherical coordinates as generalized coordinates.
13. Consider Fig. 4.13, and calculate the associated generalized forces. The disk is of mass m and the rod is of mass $2m$. There is a moment M acting on the rod at the pin joint.

SECTION 4.6

14. For the two links attached to a spring as shown in Fig. 4.31, find the equilibrium position. The spring is not stretched when the rods are horizontal.
15. Find the equilibrium position of the rod of mass m and length L sliding in the guide bars shown in Fig. 4.32. The spring is not stretched when the rod is vertical. The sliders are massless and the contact between the horizontal slider and the guide bar involves friction with coefficient μ.
16. Find the equilibrium position for the system shown in Fig. 4.33, with the middle mass equal to zero. Assume that the displacements are small, and that the springs

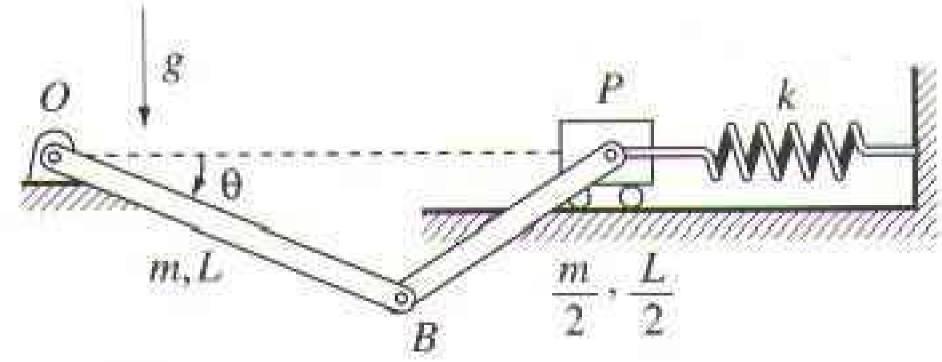

Figure 4.31

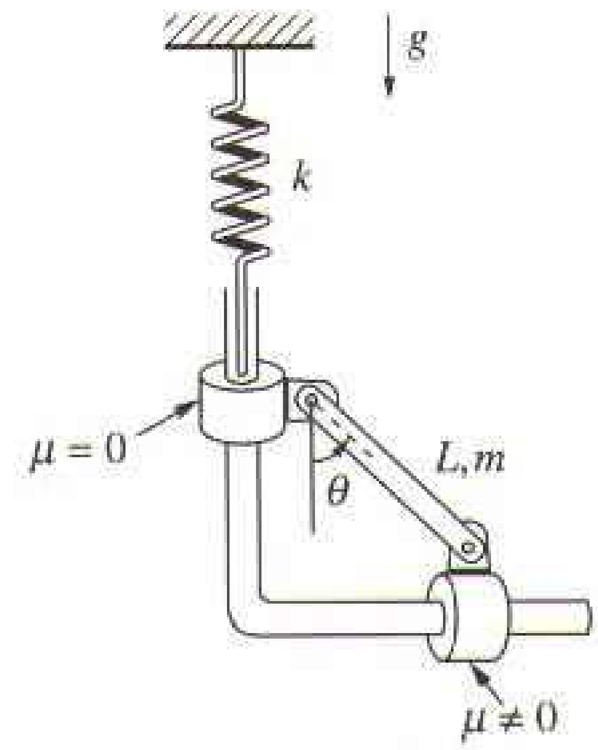

Figure 4.32

deflect only in the vertical direction. Use as generalized coordinates the translation of the center of the rod and the rotation of the rod. Then, use the deflections of the springs at A and B as generalized coordinates and obtain the equilibrium configuration. Compare the results.

17. Consider the two systems in Figs. 4.3 and 4.14 and set up the equations to find the magnitude and direction of the force **F** necessary to keep the systems at equilibrium at $\theta_1 = 30°$, $\theta_2 = 15°$.
18. Find the equilibrium position of the system in Fig. 4.34.
19. Find the equilibrium position of the pulley system in Fig. 1.77 (Problem 1.36). Use the constrained coordinate approach.

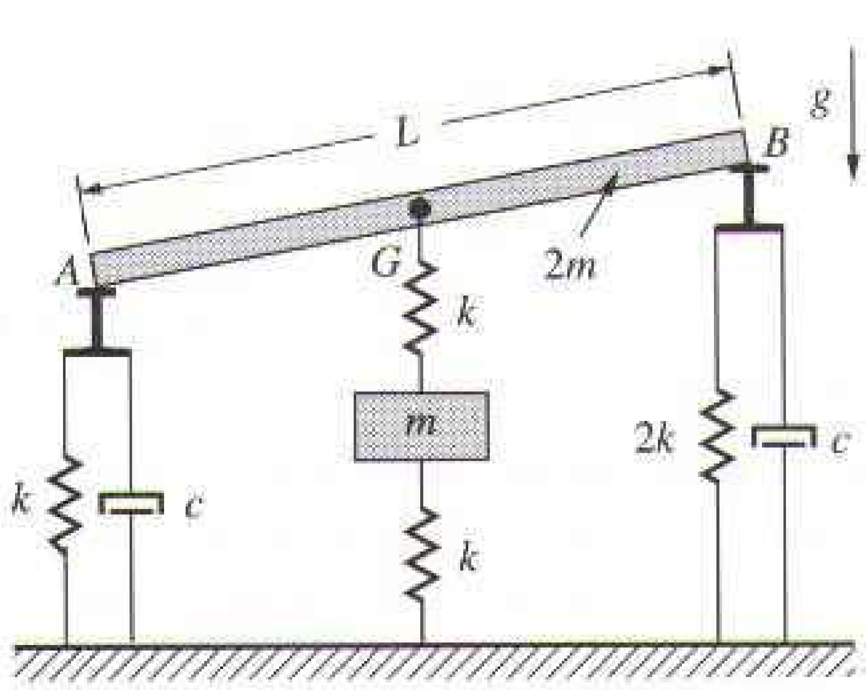

Figure 4.33

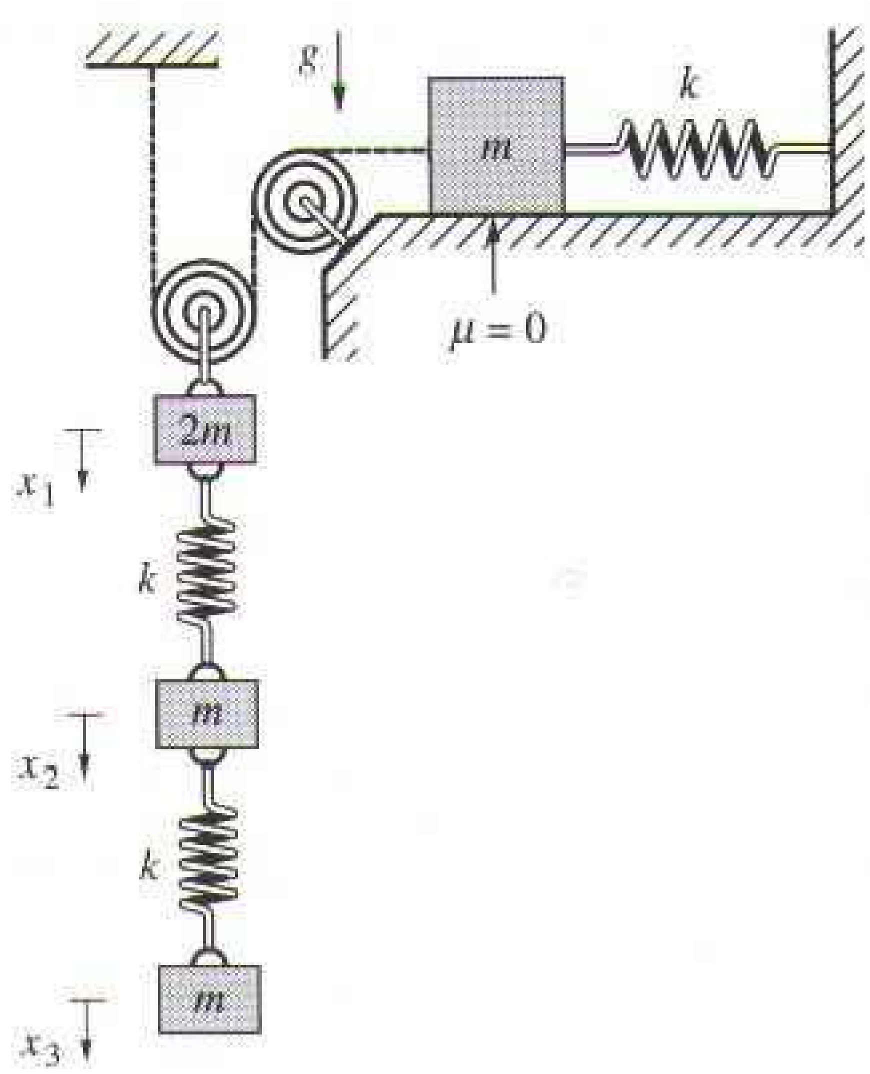

Figure 4.34

SECTION 4.7

20. Find the equation of motion of the rod in Fig. 4.32 using D'Alembert's principle.
21. Find the equations of motion of the pulley system in Fig. 3.40 using D'Alembert's principle. The pulleys are massless.

SECTION 4.8

22. Find the equation of motion of the system mechanism in Fig. 4.34 by Hamilton's principle.
23. Find the equation of motion of the rod in Fig. 4.32 by Hamilton's principle.

SECTION 4.9

24. Find the equations of motion of the system in Fig. 4.35 using Lagrange's equations.
25. Use Lagrange's equations to derive the equations of motion for the Foucault's pendulum in Chapter 2.
26. Figure 4.36 depicts a simplified illustration of a spacecraft to which a robot arm is attached at the center of mass. The robot arm moves by a moment T exerted to it at the pin joint by a motor on the spacecraft. Considering only plane motion for both the spacecraft and the robot arm, derive the equations of motion by
    a. Separating the two masses, writing force and moment balances, and eliminating constraints.
    b. Using Lagrange's equations. Compare the complexity in both cases.
27. A block of mass m and length L is positioned over a semicircular block (Fig. 4.37). It is given that friction is sufficient to prevent slipping. Derive the equation of the rod as it rocks over the semicircular block.

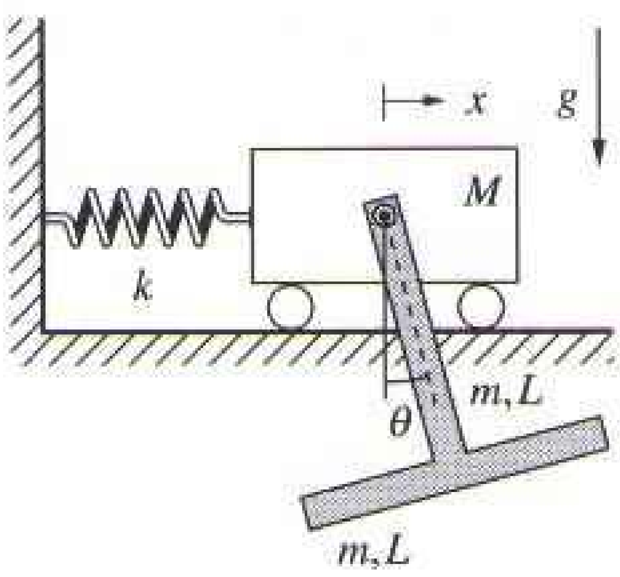

Figure 4.35

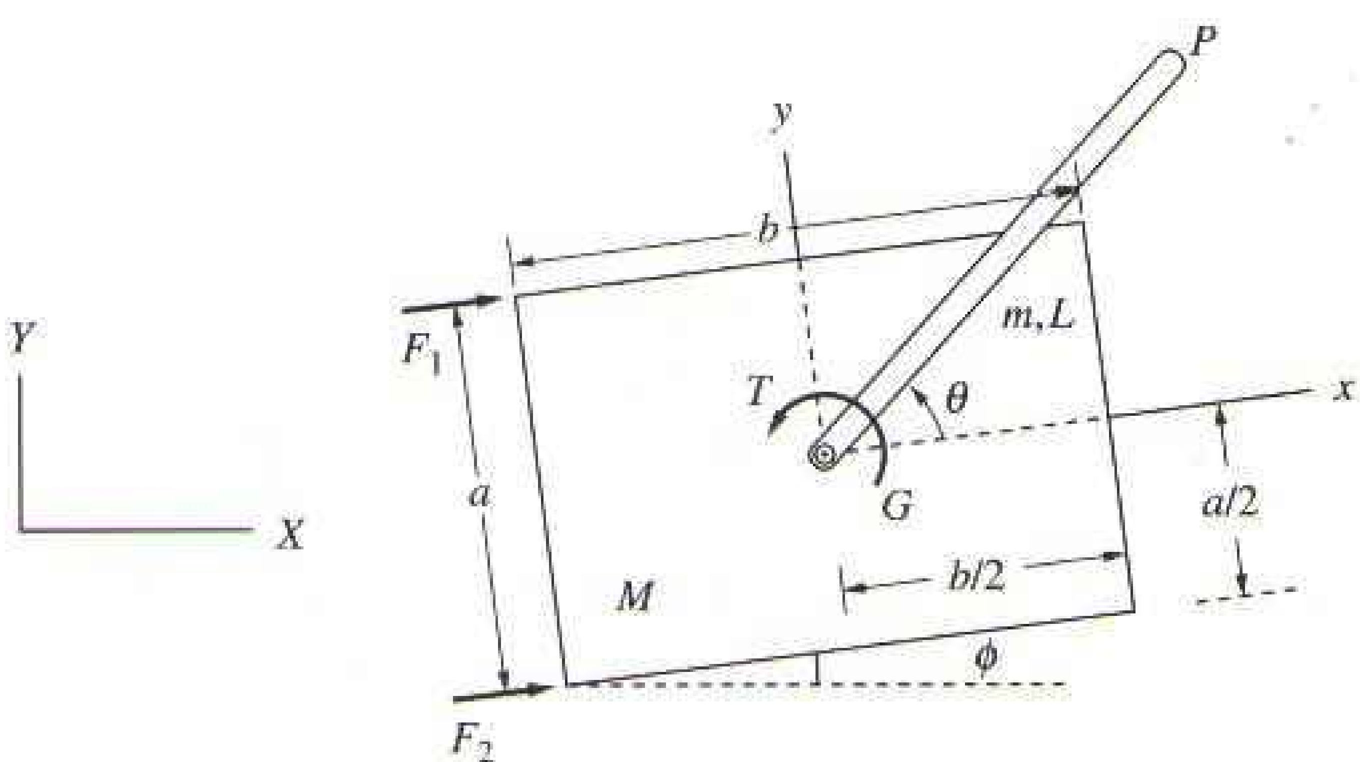

Figure 4.36

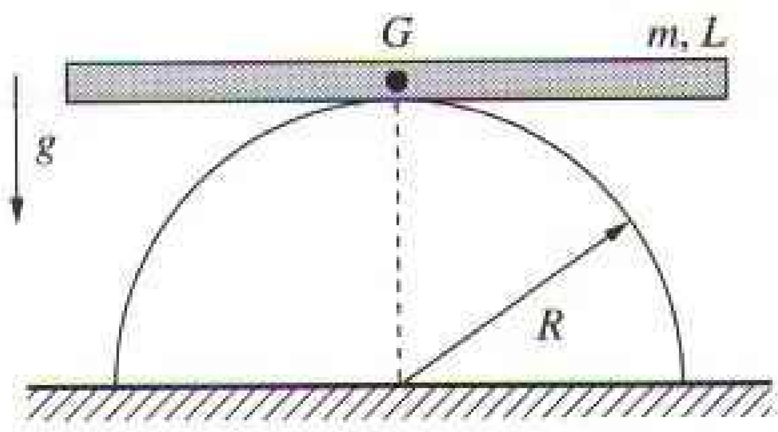

Figure 4.37

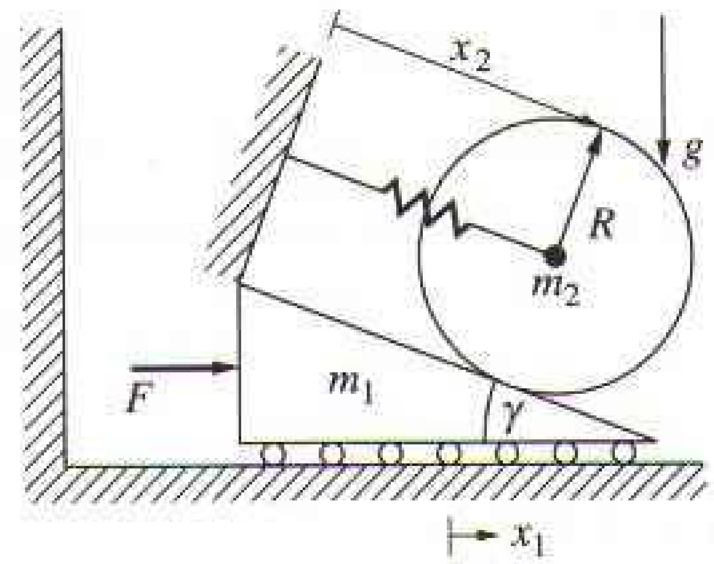

Figure 4.38

28. A cylinder of mass m_2 and radius R rolls without slipping on a wedge of mass m_1 (Fig. 4.38). The wedge is moving under the influence of the force F with no friction. Obtain the equations of motion.
29. Find the equation of motion of the rod in Fig. 4.32 using Lagrange's equations.
30. Find the equations of motion of the pulley system in Fig. 3.40 using Lagrange's equations.
31. Find the equation of motion of the bead in Fig. 4.39 sliding without friction in the parabolic tube described by $z = x^2/4$, while the tube is rotating about the z axis with constant angular velocity Ω.
32. Find the equations of motion of the system in Fig. 4.33, using Lagrange's equations. Assume small motions and that the springs and dashpots deflect only vertically.
33. A particle of mass m is constrained to slide without friction down a channel attached to a cone spinning with constant angular velocity Ω, as shown in Fig. 4.40. Derive the following:
    a. A differential equation of motion describing the motion of P using Newton's second law.
    b. The equation of motion using Lagrange's equations.

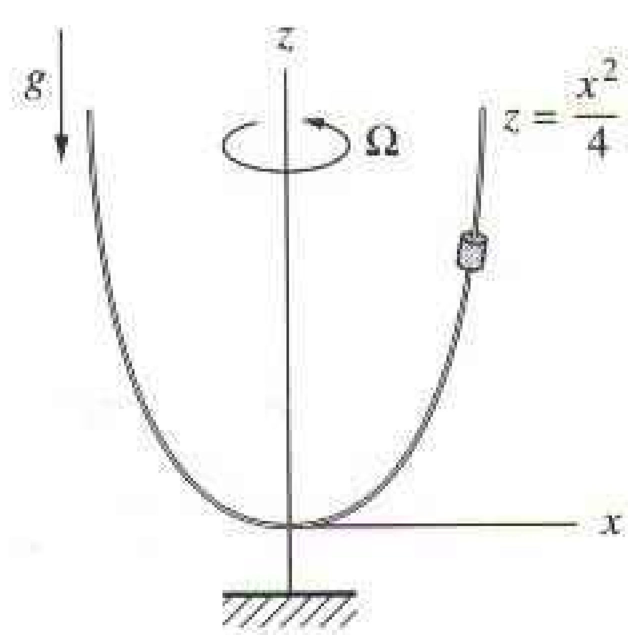

Figure 4.39

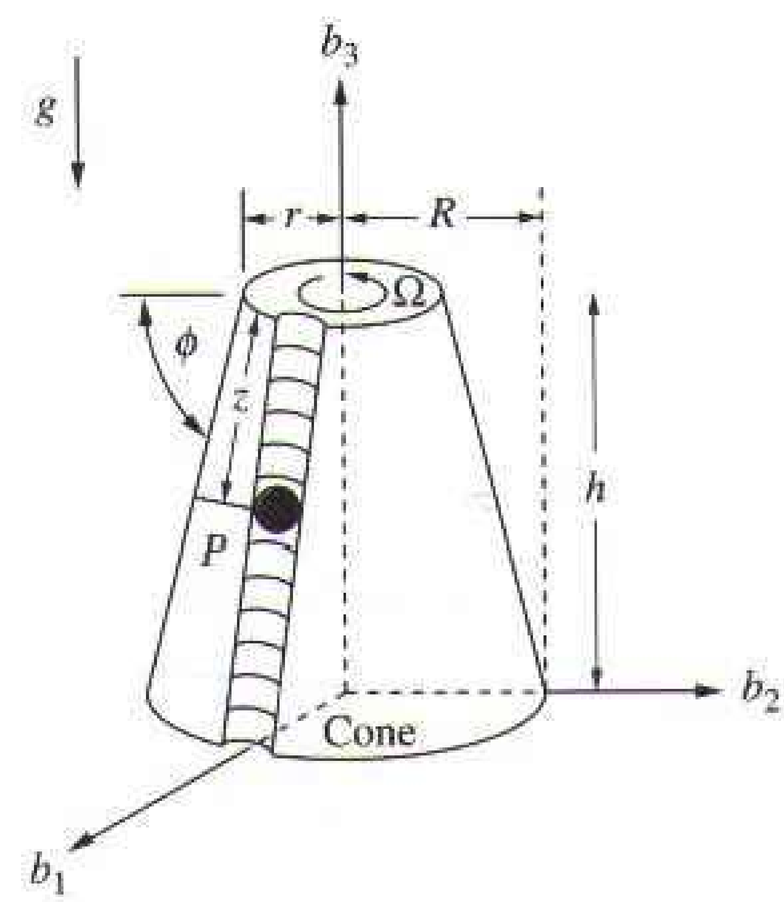

Figure 4.40

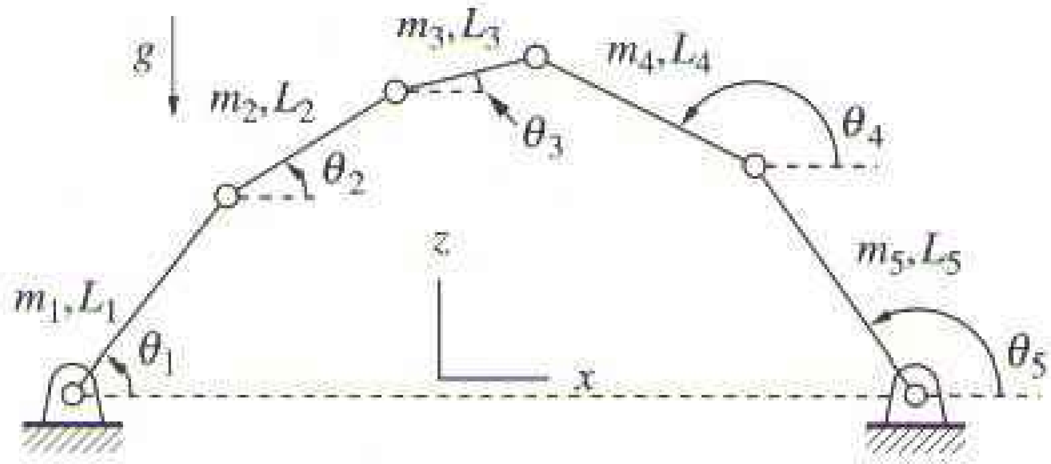

Figure 4.41

34. Consider the pendulum in Fig. 4.28 of mass m, $L_1 = L$, $L_2 = 2L$. The pendulum swings in the plane generated by the column and arm. Derive the equations of motion, using the pendulum angle θ and rotation angle ϕ of the shaft as generalized coordinates. The combined mass moment of inertia of the column and arm about the Z axis is I.
35. Consider the particle in Problem 3. Find the equations of motion.

SECTION 4.10

36. Consider the four-bar linkage mechanism in Fig. 4.24 and derive the equations of motion using constrained generalized coordinates. Consider each link separately. A moment M acts on the first link.
37. Consider the platform robot in Fig. 4.41. Derive the equations of motion using constrained coordinates for the following cases: *(a)* each link is considered separately; and *(b)* links 1 and 2 and links 4 and 5 are each considered as one system each.
38. Given the two-link system in Fig. 4.3, derive the equations of motion using the constrained coordinate formulation and using as generalized coordinates θ_1, θ_2, x_P, and y_P. Calculate the kinetic energy of the second link using the motion of its center of mass, expressed in terms of θ_2, x_P, and y_P.
39. Consider the vehicle in Fig. 4.8. We are given the constraint that the velocity of the tip of the vehicle, that is, point E, is along the x direction. Derive the equations of motion using constrained generalized coordinates.
40. Consider the pendulum in Fig. 4.28. The angular velocity of the column is kept constant at $\dot{\phi} = \Omega$ by a motor that generates a moment about the Z axis. Find the equation of motion of the pendulum and using the constraint relaxation method find the moment necessary to maintain the constant angular velocity of the column.
41. Find the equation of motion of the rod in Fig. 4.32 using the constraint relaxation method.