Vibration Mechanics Hw #2

(Free Vibration of SDOF Systems)

Issued: Tue. Mar. 05 2024

Due: Wed. Mar. 20 2024 18:00

1. Rao P. 2.7 natural frequency calculation

For small angular rotation of bar PQ about P,

$$\frac{1}{2}(k_{12})_{eq}(\theta l_3)^2 = \frac{1}{2}k_1(\theta l_1)^2 + \frac{1}{2}k_2(\theta l_2)^2$$

i.e.,
$$(k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2)/l_3^2$$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

$$k_{eq} = \frac{(k_{12})_{eq}\, k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2\right\}k_3}{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

2. Rao P. 2.16 natural frequency calculation

2.16

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

Conservation of momentum:

$$(M+m)\dot{x}_0 = mv \text{ or } \dot{x}_0 = \dot{x}(t=0) = \frac{mv}{M+m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{4k}{M+m}}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{\frac{m^2 g^2}{16 k^2} + \frac{m^2 v^2}{(M+m)\,4k}\right\}^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(-\frac{mg}{4k}\sqrt{\frac{4k}{(M+m)}}\,\frac{(M+m)}{mv}\right) = \tan^{-1}\left(-\frac{g\sqrt{M+m}}{v\sqrt{4k}}\right)$$

(a) Velocity of hammer = 15 m/s

Mass of hammer, m = 6 kg

Mass of anvil, m = 50 kg

$$A_0 = \left\{\left(\frac{6 \times 9.81}{(4)(17.5 \times 10^3)}\right)^2 + \frac{(6^2)(15^2)}{(56)(70 \times 10^3)}\right\}^{\frac{1}{2}} = 0.0082 \text{ m or } 8.2 \text{ mm}$$

$$\phi_0 = \tan^{-1}\left\{-\frac{9.81\sqrt{56}}{(15)\sqrt{70 \times 10^3}}\right\} = -1.06 \text{ degrees}$$

(b) x = 0 at static equilibrium position: $x_0 = x(t=0) = 0$. Conservation of momentum gives:

$$M\dot{x}_0 = mv \text{ or } \dot{x}_0 = \dot{x}(t=0) = \frac{mv}{M}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{\frac{m^2 v^2 (M)}{M^2\, 4k}\right\}^{\frac{1}{2}} = \frac{mv}{\sqrt{4kM}} = \frac{(6)(15)}{\sqrt{(70 \times 10^3)(50)}} = 0.048 \text{ m or } 48 \text{ mm}$$

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}(0) = 0$$

3. Rao P. 2.23 natural frequency calculation

(a) Neglect masses of rigid links. Let x = displacement of W. Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

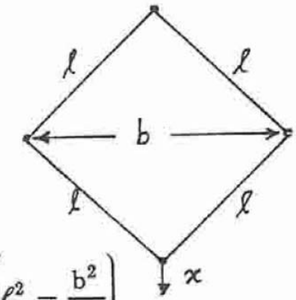
$$m\ddot{x} + k_{eq}\, x = 0$$

Natual frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2\,m}}$$

(b) Under a displacement of x of mass, each spring will be compressed by an an amount:

$$x_s = x\,\frac{2}{b}\sqrt{\ell^2 - \frac{b^2}{4}}$$



Equivalent spring constant:

$$\frac{1}{2}\,k_{eq}\,x^2 = 2\left(\frac{1}{2}\,k\,x_s^2\right)$$

or $$k_{eq} = 2\,k\left(\frac{x_s}{x}\right)^2 = 2\,k\left(\frac{4}{b^2}\right)\left(\ell^2 - \frac{b^2}{4}\right) = \frac{8\,k}{b^2}\left(\ell^2 - \frac{b^2}{4}\right)$$

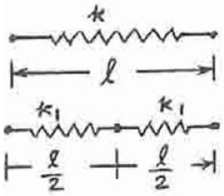
Equation of motion:

$$m\ddot{x} + k_{eq}\, x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8\,k}{b^2\,m}\left(\ell^2 - \frac{b^2}{4}\right)}$$

4. Rao P. 2.40 natural frequency calculation



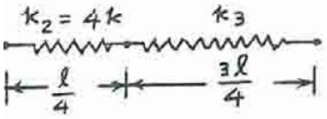
$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{total} = \frac{k_1}{2} \equiv k \; ; \; k_1 = 2k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}}$$

$$0.5 = 2\pi \sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{total}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

$$k_3 = \frac{4}{3} k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}} \quad \text{where} \quad k_{eq} = 4k + \frac{4}{3} k = \frac{16}{3} k$$

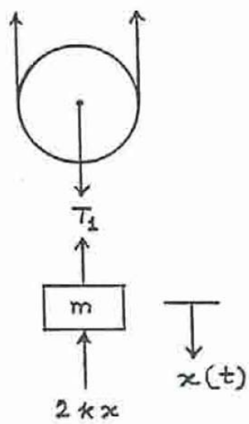
$$\therefore \tau_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4}\sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4}\left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

5. Rao P. 2.46 Equation of motion

Consider the springs connected to the pulleys (by rope) to be in series. Then

$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \quad \text{or} \quad k_{eq} = \frac{5}{6} k$$

Let the displacement of mass m be x. Then the extension of the rope (springs connected to the pulleys) = 2 x. From the free body diagram, the equation of motion of mass m:



$$m\ddot{x} + 2kx + k_{eq}(2x) = 0$$

$$\text{or} \quad m\ddot{x} + \frac{11}{3} k x = 0$$

6. Rao P. 2.87 Torsional vibration

2.67 (a) $\omega_n = \sqrt{\frac{g}{\ell}}$

(b) $m\ell^2\ddot{\theta} + ka^2 \sin\theta + mg\ell \sin\theta = 0$; $m\ell^2\ddot{\theta} + (ka^2 + mg\ell)\,\theta = 0$

$$\omega_n = \sqrt{\frac{ka^2 + mg\ell}{m\ell^2}}$$

(c) $m\ell^2\ddot{\theta} + ka^2 \sin\theta - mg\ell \sin\theta = 0$; $m\ell^2\ddot{\theta} + (ka^2 - mg\ell)\,\theta = 0$

$$\omega_n = \sqrt{\frac{ka^2 - mg\ell}{m\ell^2}}$$

configuration (b) has the highest natural frequency.

7. Rao P. 2.90 Torsional vibration

J_0 = mass moment of inertia of the ring = 1.0 kg–m^2.

I_{os} = polar moment of inertia of the cross section of steel shaft

$$= \frac{\pi}{32}\left(d_{os}^4 - d_{is}^4\right) = \frac{\pi}{4}\left(0.05^4 - 0.04^4\right) = 36.2266\ (10^{-8})\ \text{m}^4$$

I_{ob} = polar moment of inertia of cross section of brass shaft

$$= \frac{\pi}{32}\left(d_{ob}^4 - d_{ib}^4\right) = \frac{\pi}{32}\left(0.04^4 - 0.03^4\right) = 17.1806\ (10^{-8})\ \text{m}^4$$

k_{ts} = torsional stiffness of steel shaft

$$= \frac{G_s\, I_{os}}{\ell} = \frac{(80\,(10^9))\,(36.2266\,(10^{-8}))}{2} = 14490.64\ \text{N–m/rad}$$

k_{tb} = torsional stiffness of brass shaft

$$= \frac{G_b\, I_{ob}}{\ell} = \frac{(40\,(10^9))\,(17.1806\,(10^{-8}))}{2} = 3436.12\ \text{N–m/rad}$$

$k_{t_{eq}} = k_{ts} + k_{tb} = 17{,}926.76$ N–m/rad

Torsional natural frequency:

$$\omega_n = \sqrt{\frac{k_{t_{eq}}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908\ \text{rad/sec}$$

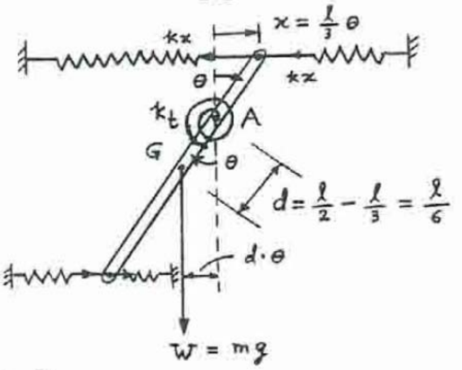
Natural time period:

$$\tau_n = \frac{2\,\pi}{\omega_n} = \frac{2\,\pi}{133.8908} = 0.04693\ \text{sec}$$

8. Rao P. 2.93 Torsional vibration

Equation of motion

$$J_A \ddot{\theta} = -W d\theta - 2k\left(\frac{l}{3}\theta\right)\frac{l}{3} - 2k\left(\frac{2l}{3}\theta\right)\frac{2l}{3} - k_t \theta$$



where

$$J_A = J_G + m d^2 = \frac{1}{12} m l^2 + m\frac{l^2}{36} = \frac{1}{9} m l^2$$

$$\therefore \frac{m l^2}{9}\ddot{\theta} + \left(mgd + 2k\frac{l^2}{9} + \frac{8kl^2}{9} + k_t\right)\theta = 0$$

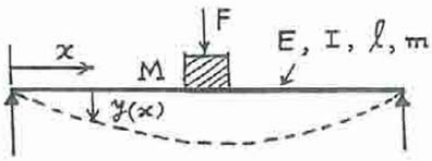
$$\omega_n = \sqrt{\frac{\left(mgd + \frac{2}{9}kl^2 + \frac{8}{9}kl^2 + k_t\right)9}{ml^2}} = \sqrt{\frac{9mgd + 10kl^2 + 9k_t}{ml^2}}$$

For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10(2000)(5)^2 + 9(1000)}{10(5)^2}} = 45.1547 \frac{\text{rad}}{\text{sec}}$$

9. Rao P. 2.106 Rayleigh method

Let m_{eff} = effective part of mass of beam (m) at middle. Thus vibratory inertia force at middle is due to $(M + m_{eff})$. Assume a deflection shape: $y(x,t) = Y(x) \cos(\omega_n t - \phi)$ where $Y(x)$ = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left(3 \frac{x}{\ell} - 4 \frac{x^3}{\ell^3}\right) \; ; \; 0 \le x \le \frac{\ell}{2}$$

where Y_0 = maximum deflection of the beam at middle = $\dfrac{F \ell^3}{48 E I}$

Maximum strain energy of beam = maximum work done by force F = $\dfrac{1}{2} F Y_0$.

Maximum kinetic energy due to distributed mass of beam:

$$= 2 \left\{ \frac{1}{2} \frac{m}{\ell} \int_0^{\frac{\ell}{2}} \dot{y}^2(x,t) \,|_{max}\, dx \right\} + \frac{1}{2} \left(\dot{y}_{max}\right)^2 M$$

$$= \frac{m\, \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y^2(x)\, dx + \frac{1}{2}\, \omega_n^2\, Y_{max}^2\, M$$

$$= \frac{m\, \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y_0^2 \left(\frac{9\, x^2}{\ell^2} + 16 \frac{x^6}{\ell^6} - 24 \frac{x^4}{\ell^4} \right) dx + \frac{1}{2}\, Y_0^2\, M\, \omega_n^2$$

$$= \frac{m\, \omega_n^2\, Y_0^2}{\ell} \left[\frac{9}{\ell^2} \frac{x^3}{3} + \frac{16}{\ell^6} \frac{x^7}{7} - \frac{24}{\ell^4} \frac{x^5}{5} \right] \Big|_0^{\frac{\ell}{2}} + \frac{1}{2}\, Y_0^2\, M\, \omega_n^2$$

$$= \frac{1}{2}\, Y_0^2\, \omega_n^2 \left(\frac{17}{35}\, m + M \right)$$

This shows that $m_{eff} = \dfrac{17}{35}\, m = 0.4857\, m$

10. Rao P. 2.130 Viscous free vibration

(i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.
(b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\dfrac{x_i}{x_{i+1}} = e^{\varsigma\, \omega_n\, \tau_d}$

$$\ln\left(\frac{x_i}{x_{i+1}}\right) = \ln 2 = 0.6931 = \frac{2\,\pi\,\varsigma}{\sqrt{1-\varsigma^2}}$$

or $39.9590\ \varsigma^2 = 0.4804$ or $\varsigma = 0.1096$

Since $\omega_d = \omega_n \sqrt{1-\varsigma^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\varsigma^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(31.6065)^2 = 5.0916\ (10^4)\ \text{N/m}$$

$$\varsigma = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

Hence $c = 2\ m\ \omega_n\ \varsigma = 2\ (\frac{500}{9.81})\ (31.6065)\ (0.1096) = 353.1164$ N–s/m

(b) From Eq. (2.135):

$$k = m\,\omega_n^2 = \frac{500}{9.81}\ (31.416)^2 = 5.0304\ (10^4)\ \text{N/m}$$

Using N = W = 500 N,

$$\mu = \frac{0.002\ k}{4\ W} = \frac{(0.002)\ (5.0304\ (10^4))}{4\ (500)} = 0.0503$$

11. Rao P. 2.140 Equation of motion

Let δx = virtual displacement given to cylinder. Virtual work done by various forces:

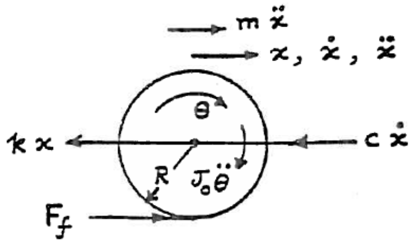
Inertia forces: $\delta W_i = -(J_0\ \ddot{\theta})\ (\delta\theta) - (m\ \ddot{x})\ \delta x = -(J_0\ \ddot{\theta})\ (\frac{\delta x}{R}) - (m\ \ddot{x})\ \delta x$

Spring force: $\delta W_s = -(k\ x)\ \delta x$
Damping force: $\delta W_d = -(c\ \dot{x})\ \delta x$
By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R}\left(\frac{\ddot{x}}{R}\right) - m\ \ddot{x} - k\ x - c\ \dot{x} = 0 \quad \text{or} \quad \frac{3}{2}\ m\ \ddot{x} + c\ \dot{x} + k\ x = 0$$

Or

Newton's second law of motion:



$$\sum F = m\ \ddot{x} = -k\ x - c\ \dot{x} + F_f \qquad (1)$$

$$\sum M = J_0\ \ddot{\theta} = -F_f\ R \qquad (2)$$

where F_f = friction force.

Using $J_0 = \frac{m\ R^2}{2}$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2\ R}\left(m\ R^2\right)\frac{\ddot{x}}{R} = -\frac{1}{2}\ m\ \ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2}\ m\ \ddot{x} + c\ \dot{x} + k\ x = 0 \qquad (4)$$

The undamped natural frequency is: $\omega_n = \sqrt{\frac{2\ k}{3\ m}}$ (5)