Vibration Mechanics Hw #3

(Harmonically Excited Vibration)

10 Points each. Total 90 Pts.

Issued: Tue. Mar. 19, 2024

Due: Fri. April 12, 2024

1. Rao P. 3.16 Forced response of an undamped system

Equivalent stiffness of wing (beam) at location of engine:

$$k = \frac{\text{force}}{\text{deflection}} = \frac{3\,E\,I}{\ell^3} = \frac{3\,E\,(\frac{1}{12}\,b\,a^3)}{\ell^3} = \frac{E\,b\,a^3}{4\,\ell^3}$$

Magnitude of unbalanced force: $= m\,r\,\omega^2 = m\,r\left(\frac{2\,\pi\,N}{60}\right)^2 = \frac{m\,r\,\pi^2\,N^2}{900}$

Equivalent mass of wing at location of engine: $M = \frac{33}{140}\,m_w = \frac{33}{140}\,(a\,b\,\ell\,\rho)$

Equation of motion: $M\,\ddot{x} + k\,x = m\,r\,\omega^2\,\sin\omega\,t$

Maximum steady state displacement of wing at location of engine:

$$X = \left|\frac{m\,r\,\omega^2}{k - M\,\omega^2}\right| = \left|\frac{\left(\frac{m\,r\,\pi^2\,N^2}{900}\right)}{\left\{\frac{E\,b\,a^3}{4\,\ell^3} - \frac{33}{140}\,a\,b\,\ell\,\rho\left(\frac{2\,\pi\,N}{60}\right)^2\right\}}\right|$$

$$= \left|\frac{m\,r\,\ell^3\,N^2}{22.7973\,E\,b\,a^3 - 0.2357\,\rho\,a\,b\,\ell^4\,N^2}\right|$$

2. Rao P. 3.25 Forced response of an undamped system

Equation of motion for rotation about O:

$$J_0\,\ddot{\theta} = -k\,\frac{\theta\,\ell}{4}\,\frac{\ell}{4} - k\,\frac{\theta\,3\,\ell}{4}\,\frac{3\,\ell}{4} + M_0\,\cos\omega\,t$$

$$\text{i.e.,}\quad J_0\,\ddot{\theta} + \left(\frac{5}{8}\,k\,\ell^2\right)\theta = M_0\,\cos\omega\,t$$

where $J_0 = \frac{1}{12}\,m\,\ell^2 + m\,(\frac{\ell}{4})^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583$ kg–m^2

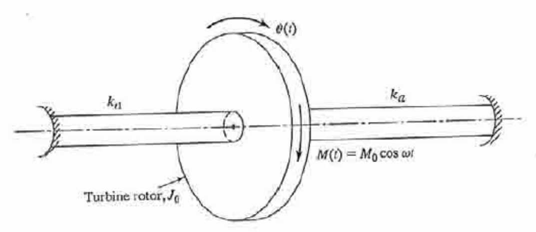
and $\omega = 1000$ rpm $= 104.72$ rad/sec. Steady state solution is:

$$\theta_p(t) = \Theta\,\cos\omega\,t$$

where

$$\Theta = \frac{M_0}{\frac{5}{8}\,k\,\ell^2 - J_0\,\omega^2} = \frac{100}{5000\,(\frac{5}{8})\,(1^2) - 1.4583\,(104.72^2)} = -\,0.007772 \text{ rad}$$

3. Rao P. 3.38 Forced response of a damped system



$$M_t = M_0 \cos \omega t \,, \quad k_t = k_{t1} + k_{t2}$$

Equation of motion:

$$J_0 \ddot{\theta} + c_t \dot{\theta} + (k_{t1} + k_{t2})\, \theta = M(t) = M_0 \cos \omega t \qquad (1)$$

For the given data, Eq. (1) becomes

$$0.05\, \ddot{\theta} + 2.5\, \dot{\theta} + 7000\, \theta = 200 \cos 500 t \qquad (2)$$

steady state response of the turbine rotor can be expressed, similar to Eqs. (3.25), (3.28) and (3.29) for a torsional system, as

$$\theta_p(t) = \Theta \cos(\omega t - \phi) \qquad (3)$$

where

$$\Theta = \frac{M_0}{\left\{(k_t - J_0 \omega^2)^2 + c_t^2 \omega^2\right\}^{\frac{1}{2}}} \qquad (4)$$

and

$$\phi = \tan^{-1}\left(\frac{c_t \omega}{k_t - J_0 \omega^2}\right) \qquad (5)$$

For the given data,

$J_0 = 0.05$, $M_0 = 200$, $k_t = 7000$, $c_t = 2.5$, $\omega = 500$

Hence Eqs. (4) and (5) give

$$\Theta = \frac{200}{\left[(7000 - 0.05 \times 25 \times 10^4)^2 + (2.5)^2 (25 \times 10^4)\right]^{\frac{1}{2}}}$$

$$= 6.2868 \times 10^{-6} \text{ rad}$$

$$\phi = \tan^{-1}\left(\frac{2.5 \times 500}{7000 - 0.05 \times 250\,000}\right)$$

$$= \tan^{-1}\left(-\frac{1250}{5500}\right) = \tan^{-1}(-0.2273)$$

$$= -12.8043^\circ = -0.2235 \text{ rad}$$

4. Rao P. 3.45 Forced response of a damped system

$$\frac{X}{Y} = \left[\frac{k^2 + c^2 \omega^2}{\left(k - m \omega^2\right)^2 + c^2 \omega^2} \right]^{\frac{1}{2}} .sp$$

$$\text{or} \quad \frac{10^{-6}}{Y} = \left[\frac{(10^6) + (10^3 (200 \pi))^2}{\left\{10^6 - \left(\frac{5000}{9.81}\right) (200 \pi)^2\right\}^2 + \left\{(10^3) (200 \pi)\right\}^2} \right]^{\frac{1}{2}}$$

$$\text{or} \quad Y = 169.5294 \, (10^{-6}) \text{ m}$$

5. Rao P. 3.47 Forced response of a damped system

$m = 100$ kg, $F_0 = 100$ N, $X_{max} = 0.005$ m at $\omega = 300$ rpm $= 31.416$ rad/sec. Equations (3.33) and (3.34) yield:

$$\omega = \omega_n \sqrt{1 - 2 \zeta^2} = \sqrt{\frac{k}{m}} \sqrt{1 - 2 \zeta^2} = 31.416$$

$$\text{or} \quad k \, (1 - 2 \zeta^2) = (100) \, (31.416^2) = 98{,}696.5056 \tag{1}$$

$$\text{and} \quad X_{max} = \delta_{st} \frac{1}{2 \zeta \sqrt{1 - \zeta^2}} = \frac{F_0}{k} \frac{1}{2 \zeta \sqrt{1 - \zeta^2}} = 0.005$$

$$\text{or} \quad k \zeta \sqrt{1 - \zeta^2} = \frac{F_0}{2 \, (0.005)} = 10{,}000.0 \tag{2}$$

Divide Eq. (1) by (2):

$$\frac{1 - 2 \zeta^2}{\zeta \sqrt{1 - \zeta^2}} = 9.8696 \tag{3}$$

Squaring Eq. (3) and rearranging leads to:

$$101.4090 \, \zeta^4 - 101.4090 \, \zeta^2 + 1 = 0 \quad \text{or} \quad \zeta = 0.0998, \; 0.9950$$

Using $\zeta = 0.0998$ in Eq. (1), we obtain

$$k = \frac{98696.5056}{1 - 2 \, (0.0998^2)} = 100{,}702.4994 \text{ N/m}$$

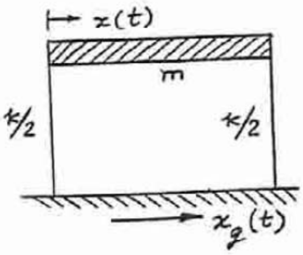
Since $\zeta = \frac{c}{2 \, m \, \omega_n}$, we find

$$c = 2 \, m \, \omega_n \, \zeta = 2 \, (100) \sqrt{\frac{100702.4944}{1000}} \, (0.0998) = 633.4038 \text{ N--s/m}$$

6. Rao P. 3.55 Response of a system under the base harmonic motion

$$\ddot{y}(t) = \ddot{x}_g(t) = A\cos\omega t \;; \qquad \dot{y}(t) = \frac{A}{\omega}\sin\omega t + B_1$$

$$y(t) = -\frac{A}{\omega^2}\cos\omega t + B_1 t + B_2$$



Assuming $y(0) = \dot{y}(0) = 0$, we get

$$y(t) = -\frac{A}{\omega^2}\cos\omega t$$

Equation of motion:

$$m\ddot{x} + k(x-y) = 0$$

i.e., $m\ddot{z} + kz = -m\ddot{y} = -m\ddot{x}_g(t) = -mA\cos\omega t$

where $z = x - y$

Solution is : $$z(t) = \frac{-mA\cos\omega t}{k - m\omega^2}$$

$$\therefore \quad x(t) = z(t) + y(t) = -\left(\frac{m}{k - m\omega^2} + \frac{1}{\omega^2}\right)A\cos\omega t$$

7. Rao P. 3.75 Rotating unbalance

Let width $= 0.5$ m and thickness $= t$ m.

$$I = \frac{1}{12}(0.5)\,t^3 = \frac{t^3}{24} \; m^4$$

$$k = \frac{3EI}{l^3} = \frac{3(2.07\times10^{11})(t^3/24)}{(5)^3} = 2.07\times10^8\, t^3 \;\; N/m$$

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\,m}}$$

where m = mass of beam $= (5\times0.5\times t)\left(\frac{76.5\times10^3}{9.81}\right) = 19495.41\,t$ kg

$$\omega_n = \sqrt{\frac{2.07\times10^8\, t^3}{75 + 0.2357\,(19495.41\,t)}}$$

$$r = \frac{\omega}{\omega_n} = 125.664\sqrt{\frac{75 + 4595.0688\,t}{2.07\times10^8\, t^3}}$$

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k\,|r^2-1|}$$

i.e., $$0.5 = \frac{5000}{(2.07\times10^8\, t^3)\left\{(125.664)^2\left[\frac{75 + 4595.0688\,t}{2.07\times10^8\, t^3}\right] - 1\right\}}$$

i.e., $1.3108\times10^4\, t^3 - 4595.069\,t - 74.367 = 0$

By trial and error, the value of t is found as

$t \approx 0.6$ m.

Since this is too large, we can start with a new width such as 1.0 m.

8. Rao P. 3.92 Coulomb damping

$m = 25$ kg, $k = 10000$ N/m, $\mu = 0.3$

$\omega = 8$ Hz $= 50.2656$ rad/s

$X = 0.2$ m

Eq. (3.88) gives

$$c_{eq} = \frac{4 \mu N}{\pi \omega n} = \frac{4\,(0.3)\,(25 \times 9.81)}{\pi\,(50.2656)(0.2)}$$

$$= 9.3183 \ \text{N-s/m}$$

Note: It is assumed that friction force is small compared to F_0 in finding the equivalent viscous damping constant of the system.

9. Rao P. 3.97 Other types of damping

Damping force = $F = c(\dot{x})^n$

Energy dissipated per quarter cycle during harmonic motion $x(t) = X \sin \omega t$ is

$$\frac{\Delta W}{4} = \int_0^{\pi/2\omega} c(\dot{x})^n \, dx = \int_0^{\pi/2\omega} c\,(\omega X \cos \omega t)^n \, dx$$

But $dx = \dot{x}\, dt = \omega X \cos \omega t \cdot dt$

$$\Delta W = 4c\,\omega^{n+1} X^{n+1} \int_0^{\pi/2\omega} \cos^{n+1} \omega t \cdot dt$$

$$= 4c\,\omega^{n+1} X^{n+1} \left\{ \frac{1}{(n+1)\omega} \cos^n \omega t \cdot \sin \omega t \Big|_0^{\pi/2\omega} + \frac{n}{n+1} \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \right\}$$

$$= 4c\,\omega^{n+1} X^{n+1} \left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt$$

Equating this expression to $\pi c_{eq} \omega X^2$, we obtain

$$c_{eq} = \frac{4c\,\omega^n X^{n-1}}{\pi} \left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \equiv c\,\omega^n X^{n-1} \alpha_n$$

where $\alpha_n = \frac{4}{\pi}\left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt$ ---- (E_1)

For example, for $n=2$, (E_1) becomes

$$\alpha_n = \frac{4}{\pi}\left(\frac{2}{3}\right) \int_0^{\pi/2\omega} \cos \omega t \cdot dt = \frac{8}{3\pi} \left(\frac{\sin \omega t}{\omega}\right)_0^{\pi/2\omega} = \frac{8}{3\pi\omega}$$

and hence $c_{eq} = \frac{8 c \omega X}{3\pi}$

which can be seen to be same as the expression found in Example 3.7.

For few other values of n, α_n can be found as follows:

n	1	2	3	4
α_n	$\frac{1}{\omega}$	$\frac{8}{3\pi\omega}$	$\frac{3}{4\omega}$	$\frac{32}{15\pi\omega}$

The amplitude can be found as

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}}$$

$$= \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c^2 \omega^{2(n+1)} X^{2(n-1)} \alpha_n^2}}$$